Probability

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1 Introduction

Many of us are probably familiar with notions of probability. For example: coin tossing problems, colored balls in bags, decks of cards, etc. This worksheet will begin to formalize our intuitions of probability.

Problem 1 *Alice and Bob have a bucket of black and white balls. They decide to play a game: each player takes a turn picking a ball at random from the bucket, and does not put it back. Whoever picks a white ball first wins. Alice goes first. What is the probability that Alice wins if there are 5 black balls and 1 white ball? What about 6 black balls and 2 white balls?*

Problem 2 *Four children were born at the hospital yesterday. If each child has an equal chance to be a boy or a girl, what is more likely to occur: 2 are girls and 2 are boys, or 3 are of one gender and 1 is of the other gender?*

Problem 3 *Chloe chooses a real number from the interval* $[0, 1000]$*. Independently, David chooses a real number from the interval* $[0, 2000]$*. What is the probability that David's number is larger than Chloe's number?*

Problem 4 *Suppose 7 fair 6-sided dice are thrown. What is the probability that the sum of the numbers on the top faces adds up to 10?*

Problem 5 *Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice? Explain why or why not. (If you are unsure, what if there were 100 doors, 99 goats, and the host opens 98 doors which the host knows are goats. Would you switch?)*

2 Formal Probability

Definition 1

1. The **Sample Space** of an experiment, denoted Ω, is the set of all the possible outcomes or results.

2. An **Event**, usually denoted by A or A_i, is a subset of Ω

3. An **Event Space**, denoted $\mathcal{F}$, is any non-empty collection of events such that if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and if $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

4. A **Probability Function**, denoted $P : \mathcal{F} \to \mathbb{R}$, is a function such that $P(A) \geq 0$ for all events $A \in \mathcal{F}$, $P(\Omega) = 1$, and if $A_1, A_2, \ldots$ are a disjoint countable collection of events, then $P(\cup A_i) = \Sigma_{i=1}^{\infty} P(A_i)$

5. A **Probability Space** is a triple $(\Omega, \mathcal{F}, P)$ as in the previous definitions

Example 1 *Suppose our experiment is flipping a fair two-sided coin once. Our sample space would then be* $\Omega = \{H, T\}$*, the set of all possible outcomes. Then, an event is a subset of* Ω*, namely* $A \in \{\{H\}, \{T\}, \{H, T\}, \emptyset\}$*. Then we can assign probabilities to each outcome,* $P(\{H\}) = 0.5$*,* $P(\{T\}) = 0.5$*,* $P(\{H, T\}) = 1$*,* $P(\emptyset) = 0$*.*

Problem 6 *Write out the sample space, events, and probability function for flipping an unfair (0.6 heads, 0.4 tails) coin three times. Give an example of an event space.*

Problem 7 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Show that* $\emptyset, \Omega \in \mathcal{F}$ *for any event space* $\mathcal{F}$*.*

Problem 8 *Suppose that* $A_1, \ldots, A_n \in \mathcal{F}$*. Show that* $\bigcap_{i=1}^n A_i \in \mathcal{F}$ *i.e. show the intersection of a finite collection of events which are in* $\mathcal{F}$ *is also in* $\mathcal{F}$*. (Hint: De Morgan's Law)*

Problem 9 *Suppose we have n bins and r distinguishable balls. We can fit as many balls as we want in each bin. We place the balls uniformly at random in the bins.*

1. *Describe Ω in words.*

2. *What is the value of $|\Omega|$? (The size of the sample space)*

3. *Let A be the event that the first bin has exactly k balls. What is $|A|$? Deduce the value $P(A)$.*

Problem 10 *Suppose we have a strange 6-sided die where* $P(<5) = 0.6$, $P(=4) = 0.2$, $P(>2) = 0.9$. *Describe 2 different probability spaces* $(\Omega, \mathcal{F}, P)$ *satisfying the conditions.*

Problem 11 *(Simpson's Paradox) At a hospital, there are two groups of 350 patients with either a small or large kidney stone. Each group was given a different treatment, Treatment A and Treatment B. In the first group of 350 patients, Treatment A had a 93% success rate on small kidney stones, and 73% on large kidney stones, and an overall success rate of 78%. In the second group, Treatment B had an 87% success rate on small kidney stones, and 68% on large kidney stones. However, the overall success rate was 83%. How is this possible?*

Problem 12 *Alice and Bob take part in a street protest against an evil dictator. They get arrested an put in jail, each in a single person cell. The guards give each of them a coin, ask to flip it and to guess the result of the other person's flip. If at least one of the prisoners guesses right, both go free right away. If they both guess wrong, they will stay in jail for ten years. Alice and Bob cannot communicate while in their cells. However, they had time to decide on a common strategy on the way to the jail. If Alice and Bob have the best strategy possible, what is their chance to go home soon?*

Problem 13 *Oleg wrote ten letters to Math Circle parents and addressed the ten envelopes. However, he left the final stages of mailing to a careless secreatry who didn't pay attention, inserting the letters into the envelopes at random. (However, she did manage to fit exactly one letter in each envelope.) What is the probability that exactly nine of the ten letters is correctly addressed?*