FALL 2022

OLGA RADKO MATH CIRCLE
ADVANCED 1
DECEMBER 4TH, 2022

1. Logic

Problem 1.1 (2 points)**.** Suppose the statement $a \to \neg b$ is false. Find all combinations of truth values of c and d such that $(\neg b \to c) \wedge (\neg a \vee d)$ is true.

Solution. We must have that $a = T$ and $b = T$. Thus, all combinations are $(c, d) \in \{(T, T), (F, T)\}$

Problem 1.2 (2 points)**.** Consider the statements $(a \wedge b) \to c$ and $(a \to c) \wedge (b \to c)$. Are they logically equivalent? If yes, prove it. If not, give a counterexample.

Solution. They are not. Counterexample: $(a, b, c) = (F, T, T)$, first is true, second is false.

Problem 1.3 (2 points)**.** Rewrite this statement using only the connectives $\neg$ and $\wedge$: $((A \Rightarrow B) \vee (\neg C \vee D))$

Solution. $\neg((A \wedge \neg B) \wedge (C \wedge \neg D))$

Problem 1.4 (4 points)**.** Let $A, B, C \in \mathcal{L}$. Suppose that $\{(\neg A \vee B), (B \Rightarrow C), A\} \subseteq \Sigma$ where Σ is an $\mathcal{L}$-theory. Prove that $\Sigma \vdash C$. (Reminder, this means from the set $\{(\neg A \vee B), (B \Rightarrow C), A\}$, deduce C using valid sentences or Modus Ponens)

Solution. $\sigma_1 = (\neg A \vee B)$ $\sigma_2 = (B \Rightarrow C)$ $\sigma_3 = A$ $\sigma_4 = (\neg A \vee B) \to (A \to B)$ $\sigma_5 = (A \to B)$ $\sigma_6 = B$ $\sigma_7 = C$

Problem 1.5 (2 points)**.** Looking for the City of Knights on the Island of Knights and Liars, a tourist came to a place where the road forked. She knew that one of the roads at the fork was going to the City of Knights while the other was going to the City of Liars. There was no sign at the fork to point the tourist in the right direction. Fortunately, there was an islander passing by. He seemed to be in a hurry, so the tourist had time for only one question. What question should the tourist ask to figure out her way?

Solution. If you came from the other city, which road would you say is the city of knights? If he comes from liars, then they would point to the city of liars since he would lie. If he comes from knights, then he would pretend to be a liar and point to the city of liars. Thus, the other is the city of knights.

Problem 1.6 (2 points). Three turtles crawl one after another in the same direction through a narrow straight pipe. The first turtle looks around and makes the following statement.
"There are no turtles in front of me and there are two turtles behind me."
The second turtle looks around and makes the following statement.
"There is one turtle in front of me and there is one more turtle behind me."
The third turtle looks around and makes the following statement.
"There is one turtle in front of me and there is one more turtle behind me."
How can you explain it?

Solution. The third turtle is lying.

2. Juggling

Problem 2.1 (3 points). Decide if the following are valid juggling patterns.

(1) 234
(2) 98
(3) 513

Solution. Yes no no

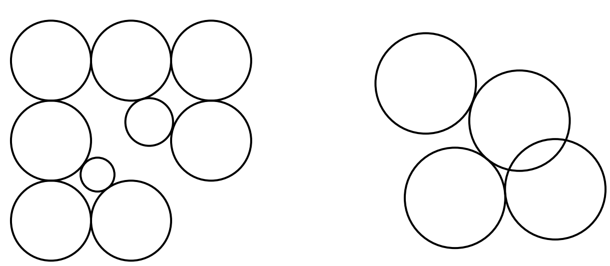
Problem 2.2 (3 points). Suppose you just threw 51515. Find a way to smoothly transition into 531531531.

Solution. You can throw 12531531... or 13153153...

Problem 2.3 (10 points). Find 3 objects (erasers, socks, etc.) and juggle a 333 pattern for at least 4 seconds. Safety first! Do not juggle sharp objects like pencils and pens.

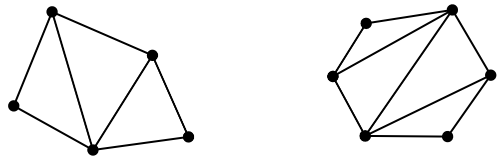
3. Graph Theory and Circle Packing

Problem 3.1 (3 points). Determine if the following pictures are circle packings. If so, draw its nerve.



Solution. Yes, no

Problem 3.2 (4 points). Draw circle packings whose nerves are the following planar graphs.



Problem 3.3 (4 points)**.** Describe the stereographic projection of a rectangle. That is, if you take a rectangle on the plane and project it onto the sphere, describe the shape you get in general. Your description must be in full, meaning that if we take something satisfying what you describe and project it back, we always get a rectangle in the plane.

Solution. Recall that parallel lines project into circles tangent at the north pole. We will get two pairs of these. The great circles tangent to each pair respectively form a right angle at the north pole.

4. Sets and Cardinality

Problem 4.1 (4 points)**.** How many functions are there with the following domain/codomain? If it is finite, write explicitly how many. If countable, write $|\mathbb{N}|$. Else, write uncountable:
1. $f : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$ for fixed $m, n \in \mathbb{N}_{>0}$
2. $f : \mathbb{N} \to \mathbb{N}$
3. $f : \{0, 1\} \to \mathbb{N}$
4. $f : \mathbb{N} \to \{0, 1\}$

Solution. 1. m^n
2. Uncountable
3. Countable
4. Uncountable

Problem 4.2 (3 points)**.** Give an explicit bijection $f : [0, 1) \to (0, 1)$

Solution. $f(0) = 1/2$, $f(1/n) = 1/(n + 1)$ for integers $n \geq 2$, $f(x) = x$ else.

Problem 4.3 (2 points)**.** Consider the set of grammatically valid finite-length sentences of any language with a countable vocabulary (for example, English). Is this set finite, countable, or uncountable? (Notice this set contains all mathematical statements)

Solution. Countable. For example, take the union of all length 1 sentences, most countable, then 2, 3, etc.

Problem 4.4 (2 points)**.** Does there exist a set A such that $A = \{1, 3, 4, |A|\}$? Does there exist a set B such that $B = \{1, 3, |B|\}$?

Solution. Let $|A| = 3$, then $A = \{1, 3, 4\}$. Not possible for $|B|$.

5. Hat Puzzles and Choice

Problem 5.1 (4 points)**.** Recall the finite hat puzzle in which n people have hats placed on their heads, and on each hat a number from 1 to n is written. All players have to guess the number on their head at the same time.

As a team, recall the solution. Then, come up to the instructors, and we will give your team "hats." Your job is to execute the strategy correctly. We will run the puzzle twice. If both times, at least one person guesses correctly, you get points for this question.

Problem 5.2 (4 points)**.** Recall the finite hat puzzle in which n people have hats placed on their heads, and on each hat a number from 1 to n is written. You can feel free to order yourselves, and then you will guess in that order.

As a team, recall the solution. Then, come up to the instructors, and we will give your team "hats." Your job is to execute the strategy correctly. We will run the puzzle twice. If both times, at most one person guesses incorrectly, you get points for this question.

Problem 5.3 (4 points)**.** Determine if the following relations are equivalence relations. If they are, give an example of a choice function.

(1) The relation "is a friend of" on the set of Facebook users.
(2) The relation $\sim$ on $\mathbb{R}$ defined by $x \sim y$ if and only if $\lfloor \frac{x}{10} \rfloor = \lfloor \frac{y}{10} \rfloor$.

Solution. No, yes. The equivalence classes are $\{[x, x+10) : x \in 10\mathbb{Z}\}$, so from each equivalence class just pick the least element or anything like that. 1 point for yes/no, 2 for correct choice function.

6. Miscellaneous

Problem 6.1 (2 points)**.** Without using a calculator, find the square root of the number 12345678987654321

Solution. 111111111

Problem 6.2 (2 points)**.** There are 100 soldiers in a company. Every four hours, three soldiers go on sentry duty. Is it possible to arrange it in such a way that after some period of time every soldier has been on sentry duty with another soldier exactly once? If you think it is possible, please show how. If you think it is not possible, please explain why not.

Solution.

Problem 6.3 (2 points, HMMT 2021)**.** A perfect power is an integer n that can be represented as a^k for some positive integers $a \geq 1$ and $k \geq 2$. Find the sum of all prime numbers $0 < p < 50$ such that p is 1 less than a perfect power.

Solution. 41.

Problem 6.4 (2 points, HMMT 2021)**.** Let p, q, r be primes such that $2p + 3q = 6r$. Find $p + q + r$.

Solution. 7.

Problem 6.5 (2 points, HMMT 2021)**.** Squares $ABCD$ and $DEFG$ have side lengths 1 and 1/3, respectively, where E is on CD and points A, D, G lie on a line in that order. Line CF meets line AG at X. Find the length of AX.

Solution. 3/2.

Problem 6.6 (2 points)**.** Find a real number x that solves $\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-x}}}} = x$.

Solution. $\frac{1+\sqrt{13}}{2}$