ORMC Olympiad Group
Fall: Week 8
Geometry: Circles

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Problems

1. Let O be the center of the circle Γ with radius 150. Let P be a point outside of the circle such that $OP = 250$. The tangents PT and PS are drawn so that T and S are on the circle. Point Q is chosen on the segment PT such that $SQ \perp TP$. SQ intersects with the circle S again at point X. What ia the length of the segment QX.

2. Let ABC be acute triangle and AH is an altitude with $H \in [BC]$. $AH = 4, BH = 3, HC = \frac{1}{3}$, and the circumcircle of AHC cuts the side AB at D. Chose K on the side BC so that $BK = 1$, and DK intersects the circumcircle of AHC again at L. The length of KL can be written as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers, b is square-free and $(a, c) = 1$. What is $a + b + c$?

3. **(9-point Circle)** Let ABC be a triangle with orthocenter H. Let H_a, H_b and H_c be feet of the altitudes from vertices A, B, C to the opposide sides. Let M_a, M_b and M_c be midpoints of the sides BC, AC and AB. Let X, Y and Z be midpoints of segments AH, BH and CH. Prove that said 9 points $H_a, H_b, H_c, M_a, M_b, M_c, X, Y, Z$ lie on a circle. This circle is called $9 - \textit{point circle}$.

Additional Question: What can you say about center of this circle? How about radious?

4. **(Simpson's Line)** Let ABC be a triangle. P is a point on the plane of ABC. Let X, Y, Z be the feet of the perpendiculars from point P to the lines BC, AC and AB respectively. Then following are equivalent
   (a) P lies on the circumcircle of ABC
   (b) X, Y, Z are collinear.
5. **(LAMC 2008 - modified)** Let AB and CD be two diameters of a circle with center O and radius 12. Assume $\angle AOC = 45°$. Let M be a point on the circle, and points P and Q are chosen on the diameters AB and CD so that $\angle MQA = \angle MPB = 60°$. Compute PQ.
6. (AIME-II 2013 8 - modified) A hexagon that is inscribed in a circle has side lengths 10, 10, 20, 10, 10, and 20 in that order. The radius of the circle can be written as $p + q\sqrt{r}$, where p, q and r are positive integers, r is square-free integer. Find $p + q + r$.
7. **(Turkey JBMO TST-2013)** Let D be a point on the side BC of an equilateral triangle ABC where D is different than the vertices. Let I be the excenter of the triangle ABD opposite to the side AB and J be the excenter of the triangle ACD opposite to the side AC. Let E be the second intersection point of the circumcircles of triangles AIB and AJC. Prove that A is the incenter of the triangle IEJ.
8. **(IMO 2018)** Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.
9. **(Prasolov)** Given circle S, points A and B on it and point C on chord AB. For every circle S' tangent to chord AB at point C and intersecting circle S at points P and Q consider the intersection point M of lines AB and PQ. Prove that the position of point M does not depend on the choice of circle S'.
10. **(USAJMO 2012)** Given a triangle ABC, let P and Q be points on

segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

11. **(USA TST 2004)** Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on AD. Let ω_1 and ω_2 intersect side BC at E and F, respectively. Denote by X the intersection of DF, AB and Y the intersection of DE, AC. Show that $XY \parallel BC$.