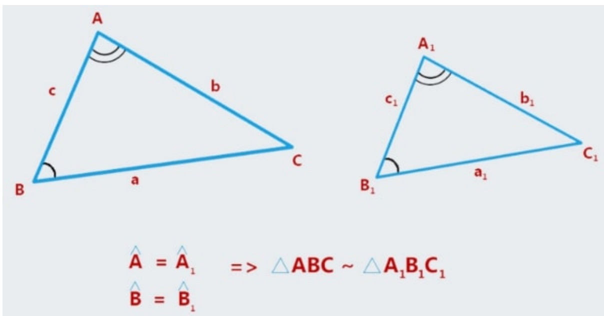
ORMC AMC Group: Week 8
Geometry: Similar Triangles

November 20, 2022

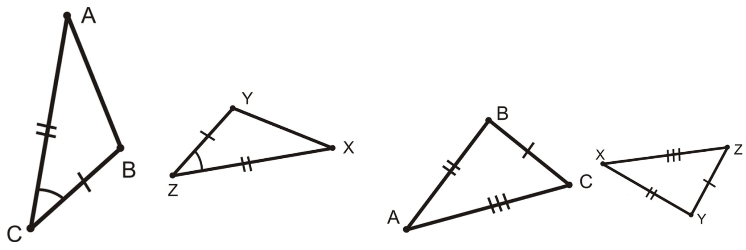
1 Similar Triangles

Two triangles are considered *similar* when one can be obtained from the other by (uniform) scaling, translation, and rotation. That is, the triangles can be considered to have the same shape, but not the same size. If triangles ABC and DEF are similar triangles, we write $ABC \sim DEF$.

In particular, the angles of two triangles must be the same for the triangles to be similar. Note that since the angles always sum to 180°, the third angle is completely defined by the first two. So, it is sufficient to say that two triangles are similar when *two* corresponding angles are the same. This is called AA or AAA similarity.



There are a couple other ways to show two triangles are similar. Recall the triangle congruence theorems: SSS, SAS, ASA, AAS. All of these are different, but in terms of similar triangles, AAS and ASA are already covered by AA. For congruence, an "S" represents the statement: "sides have the same length." However, for similarity, it represents "sides are in the same proportion."



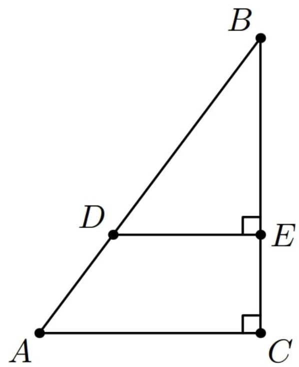
Once you have shown that two triangles are similar using one of the methods written above, the main use of similarity is that *the lengths of corresponding sides are in the same proportion.* For example, in the diagrams above,

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}.$$

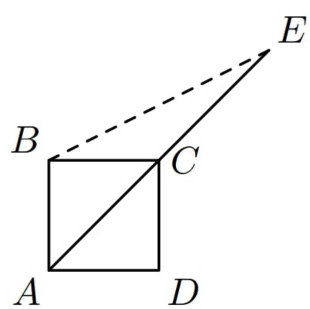
It is easiest to illustrate how useful this can be through some examples.

1.1 Examples (Basics)

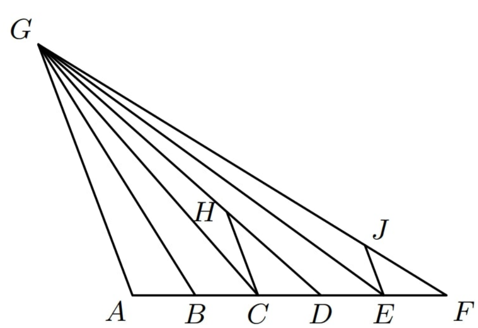
1. **(AHSME 1995)** In $\triangle ABC$, $\angle C = 90°, AC = 6$ and $BC = 8$. Points D and E are on AB and BC, respectively, and $\angle BED = 90°$. If $DE = 4$, what is the length of BD?



2. The area of square $ABCD$ is 1. As shown below, diagonal AC is extended its own length to point E. How long is BE?



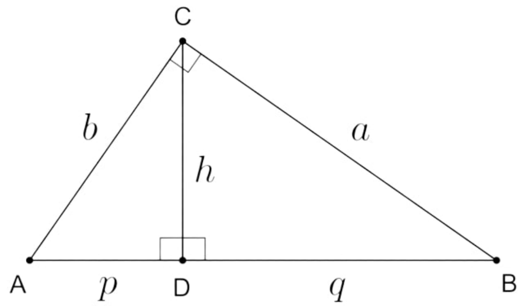
3. **(2002 AMC 10A #20)**Points A, B, C, D, E, and F lie, in that order, on AF, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on GD, and point J lies on GF. The line segments HC, JE, and AG are parallel. Find HC/JE.



1.2 Examples (Applications)

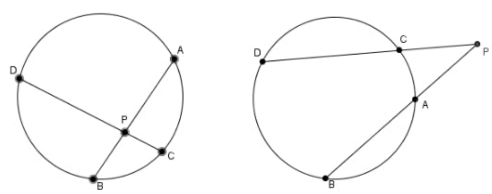
1. **Geometric Mean Property of Right Triangles**

   For a right triangle labeled as below, show that $h = \sqrt{pq}$. Additionally, show that $b = \sqrt{cp}$ and $a = \sqrt{cq}$.

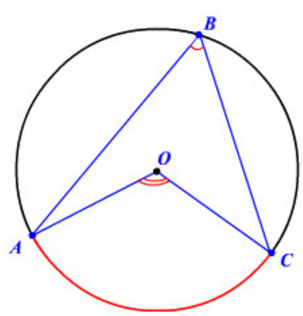


2. **Power of a Point**

   Show that in both diagrams below, $PC \cdot PD = PA \cdot PB$. (This will require 2 separate proofs).

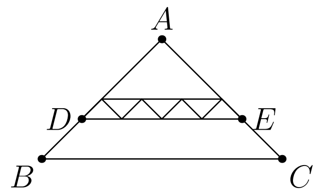


   (a) Hint: Use the **inscribed angle theorem**, which states that no matter where B is located on the major arc $\overset{\frown}{AC}$ (the longer segment connecting A to C, shown in black below), we will always have $\angle AOC = 2\angle ABC$. You can show this by using isosceles triangles and the fact that the angles of a triangle always sum to 180°.

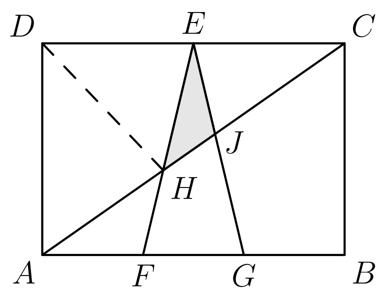


1.3 Exercises

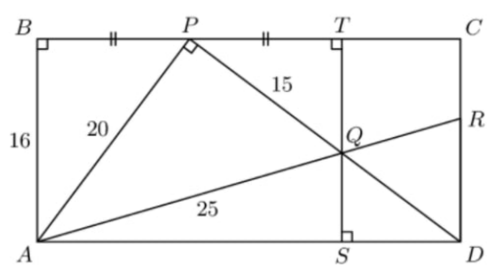
1. **(2018 AMC 10A #9)** All of the triangles in the diagram below are similar to isosceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



2. **(2001 AMC 12 #22)** In rectangle $ABCD$, points F and G lie on AB so that $AF = FG = GB$ and E is the midpoint of $\overline{DC}$. Also, $\overline{AC}$ intersects $\overline{EF}$ at H and $\overline{EG}$ at J. The area of the rectangle $ABCD$ is 70. Find the area of triangle EHJ.



3. In rectangle $ABCD$, P is a point on BC so that $\angle APD = 90^\circ$. TS is perpendicular to BC with $BP = PT$ as shown. PD intersects TS at Q. Point R is on CD such that RA passes through Q. In $\triangle PQA$, $PA = 20$, $AQ = 25$, and $QP = 15$. Find SD.



4. **(2015 AIME I #7)** In the diagram below, $ABCD$ is a square. Point E is the midpoint of $\overline{AD}$. Points F and G lie on $\overline{CE}$, and H and J lie on $\overline{AB}$ and $\overline{BC}$, respectively, so that $FGHJ$ is a square. Points K and L lie on $\overline{GH}$, and M and N lie on $\overline{AD}$ and $\overline{AB}$, respectively, so that $KLMN$ is a square. The area of $KLMN$ is 99. Find the area of $FGHJ$.