AMC 8 Training - Algebra

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1 Introduction

Today, we will go over one of the most common problem types in all AMC tests: Algebra. A simple definition of algebra is representing problems in the form of mathematical expressions. Symbols are used to represent unknown quantities.

1.1 Variables

What are variables?
Variables are symbols that stand for numbers we don't know yet. The most common variables are the letters x, y, and z. When a variable is multiplied by a number, it is represented as $4x$ and not $4 \times x$, because x is a common variable. Try this simple problem and solve for x!

Problem 1. If $x + 3 = 6$, what is x?

1.2 Solving Equations

The Golden Rule of Algebra: Do the same the same to both sides.
You can add, subtract, multiply, and divide the equation anything, as long as you do the same thing to both sides.

For example, in the following equation, you can multiply both sides by 4 to solve for y.

$$\frac{1}{4}y = 5$$

Try this practice problem:
Problem 2. Three times a number is seven more than double the number. What is the number?

Here's a problem from an actual AMC 8 test in 2020 involving algebra!

Problem 3. Luka is making lemonade to sell at a school fundraiser. His recipe requires 4 times as much water as sugar and twice as much sugar as lemon juice. He uses 3 cups of lemon juice. How many cups of water does he need?
(A) 6 **(B)** 8 **(C)** 12 **(D)** 18 **(E)** 24

2 Systems of Equations

Systems of equations are a large part of algebra. Like the name implies, there are 2 or more related equations that you can solve together. They contain the same variables, usually x and y. The number of variables and the number of equations must be the same to guarantee a single solution. Try solving this system for x and y!

$$x + y = 5$$
$$y = x + 1$$

There are 2 main ways to solve systems of equations: *substitution* and *elimination*. Today, we are only applying these problem-solving techniques with *2-variable systems*, but they can be generalized to systems of any size!

2.1 Elimination

What is elimination?
Elimination is a method of solving systems of equations where you add or subtract the equations to cancel out a variable. Elimination depends on one important part of systems of equations: **you can add and subtract the equations.** For example, we can solve for x in the following system of equations by adding the two equations!

$$3x + y = 5$$
$$-2x - y = 8$$

By adding the two equations, we can see that $x = 13$! However, real problems are almost never this simple. Most of the time, we need to multiply one or both of the equations by an integer to eliminate a variable when we add or subtract them.

Try solving this problem by eliminating x!

$$-5y + 4x = 49$$
$$7y + 2x = -20$$

Now, let's do a few practice problems:

Problem 4. A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of 34 points, and the Cougars won by a margin of 14 points. How many points did the Panthers score?

(A) 10 (B) 14 (C) 17 (D) 20 (E) 24

Problem 5. My father's age 5 years ago plus twice my age now gives 65, while my age 5 years ago plus three times my father's age now gives 130. What is my father's age? **(A)** 35 **(B)** 40 **(C)** 45 **(D)** 50 **(E)** 55

2.2 Substitution

What is substitution?
Substitution is a different approach for solving systems of equations. This involves looking at one of the equations and isolating one of the variables, then plugging it in to the other equation to solve!

Here's an example! Let's plug in x from the second equation into the first equation to solve!

$$\begin{aligned} 3y + x &= 10 \\ x &= y + 6 \end{aligned}$$

Like elimination, the problem is often not this easy. Most of the time, we must manipulate one of the equations to isolate a variable. Try solving this system with substitution!

$$\begin{aligned} 5x + y &= 20 \\ 10x - 2y &= 50 \end{aligned}$$

Now, here are a few practice problems for substitution:

Problem 6. Cadence has a collection of 52 dolls that all have either blue-eyes or green-eyes. Cadence has 16 more blue-eyed dolls than green-eyed dolls. How many green-eyed dolls does Cadence have?
(A) 8 **(B)** 12 **(C)** 16 **(D)** 18 **(E)** 24

Problem 7. You are selling hot dogs and sodas. Each hot dog costs 1.50 dollars and each soda costs 0.50 dollars. You made a total of 78.50 dollars. You sold a total of 87 hot dogs and sodas. How many hot dogs were sold and how many sodas were sold?

Problem 8. Two years ago, Gene was nine times as old as Carol. He is now seven times as old as she is. How many years from now will Gene be five times as old as Carol? **(A)** 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 10

2.3 Other Important Concepts

Substitution elimination are the 2 best ways to solve a system of equations, and you can use either of them, depending on what's most convenient.
Keep in mind that some systems of equations don't have a single value for x and y that satisfy them. If the two equations can be combined to make an equation that is never true, there is no solution. For example, the following system has no solutions that work:

$$x + 2y = 8$$
$$x + 2y = 5$$

We can see that by setting the $x + 2y$ parts equal to each other, we get 5=8.
We can also get an infinite number of solutions. If the two equations can be combined to make an equation that is always, true, there will be an infinite number of solutions. For example, this system has an infinite number of solutions.

$$2x + 4y = 10$$
$$x + 2y = 5$$

The first equation is double the second, and we can get $10 = 10$ by doubling the second equation and then substituting. Finally, although we are not going there today, a common method for solving larger systems of equations is to use elimination by adding and subtracting equations to reduce the problem down to a 2-variable system, then solve it normally.

3 Polynomials

What are polynomials?
Polynomials are expressions with many terms, which usually are different powers of the same variable. Polynomials have a **degree**, which means the highest power that any term has. Today, we will mainly be focusing on quadratic polynomials, which have a degree of 2.

3.1 Quadratic Equations

Quadratic equations are usually in standard form:

$$y = ax^2 + bx + c$$

Quadratic equations have solutions, or zeroes, which means possible x-values when y is equal to 0. There are 3 strategies to find the solutions of a quadratic equation: **factoring**, **completing the square**, and **the quadratic formula**.

First, quadratic equations can often be factored into a form with two binomials of degree 1 that make them easier to solve:

$$y = x^2 + 5x + 6$$
$$y = (x + 3)(x + 2)$$

By setting y to 0, we find that the two values of x that make the solution 0 are -3 and -2. We can generalize this for any $y = (x+p)(x+q)$, where -p and -q are the solutions. By expanding this, we can express most quadratics in the form $y = x^2 + (p + q)x + pq$.

Problem 9. Try using this to find the solutions of $x^2 - 5x - 24 = y$!

This method can also be applied when the coefficient of the x^2 term is not 1, such as in the equation $2x^2 - 3x - 35 = 0$.
Here, we know that one of the binomials must be in the form $(2x + B)$ and the other must be in the form of $(x+D)$. This expands into $2x^2 + (B+2D)x + BD$. From this, we know that $BD = 35$ and $2D + B = -3$. Try some values of B and D to find the solution!

Problem 10. Factor the quadratic $y = 10x^2 - 31x - 14$. (You don't have to find the solutions)

Another way to solve quadratics is **completing the square**. Like the name implies, this involves the square of binomials. Let's find a formula for this.

Problem 11. Expand $(Ax + B)^2 = 0$

If we had a perfect square like this, we could just take the square root of both sides to find x. Completing the square allows us to do just that. Here's an example:

In the equation $x^2 + 6x + 5 = 0$, we can see that it is quite close to the perfect square $x^2+6x+9 = (x+3)^2$. We can write it like this: $x^2+6x+9-4 = (x+3)^2-4 = 0$. Adding 4 to both sides and square-rooting, we get $(x+3) = \pm 2$, and x is equal to -1, and -5.
In general, if we have the expression $x^2 + bx$, we can add $(b/2)^2$ to complete the square and get $(x + b/2)^2$.

Problem 12. Try completing the square on the equation $0 = u^2 - 15u - 14$.

Finally, when factoring does not work, there is a *quadratic formula* that can also be used to solve quadratic equations which can be derived from the form $y = ax^2 + bx + c$. Here is the formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

By plugging in a,b, and c from the standard form, we can find the two solutions to a quadratic equation. Be careful! When using the quadratic formula, remember that there are only real solutions if the term $b^2 - 4ac$ is non-negative.

Problem 13. Try using the formula on this quadratic equation: $y = 2x^2 + 20x + 42$

Finally, here are some practice quadratic equations you can solve with any of these three methods!

Problem 14. $19x = 7 - 6x^2$
Problem 15. $y^2 = 11y - 28$
Problem 16. $z^2 - 16z + 61 = 2z - 20$

3.2 Common Algebraic Identities

Finally, let's look at some common identities of polynomials. Try finding the result of $(a + b)^2$!
To do this, we can expand the polynomial by distributing the a and then the b. $(a+b)^2 = (a+b)(a+b) = a^2+ab+ba+b^2 = a^2+2ab+b^2$ This result is called the square of a sum. Try finding $(a - b)^2$ with the same method! By using these identities, we can easily factor more complicated expressions, as well as recognize them and simplify much harder problems.
Let's do one more! Expand the expression $(a + b)(a - b)$! This result is called the difference of squares.
Although we don't have time to derive all of them, here is a list of other common identities that show up in the AMCs.

Cube of a Binomial
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a - b)^3 = a^3 - 3a^2b + 3ab^2 + b^3$
Square of a Trinomial
$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
Sum of Cubes
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Cubes
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Product of Two Binomials
$(x + a)(x + b) = x^2 + (a + b)x + ab$