Geometry of Graphs II - Graphs and Circle Packings

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ORMC Advanced 1 - Fall 2022

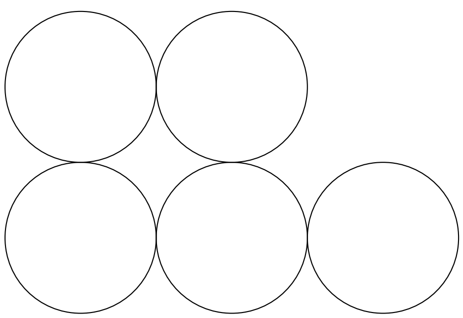
1 Circle Packings

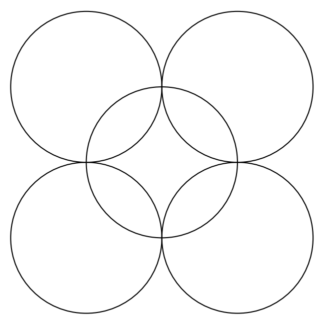
Definition 1 *A (finite)* ***planar circle packing*** *consists of a (finite) collection of circles in the plane which don't overlap (but can be tangent), and whose union is connected.*

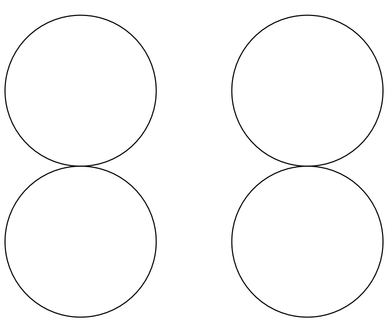
Connectedness means the same as it did for graphs, so we can think of a connected figure as one that you could draw without lifting your pencil from the page. Note that the circles don't have to be the same size!

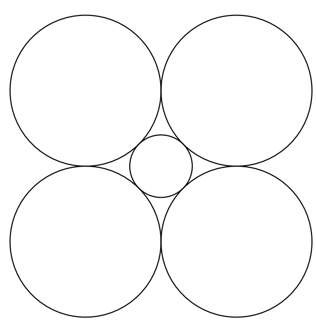
(It is also possible to have an infinite circle packing, but we will not be concerned with these for now.)

Problem 1 *The following pictures show circles in the plane. Which ones are circle packings?*





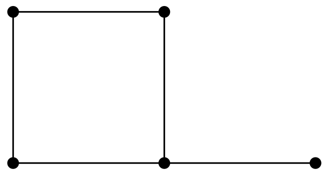
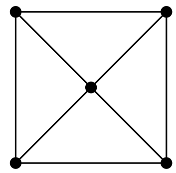




Solution: The top left and bottom right are. The bottom left is not connected, and the top right contains overlaps.

Definition 2 *The* ***nerve*** *of a (finite) circle packing is the graph whose vertices are the centers of the circles in the packing, with two such vertices connected exactly when the two corresponding circles are tangent.*

Problem 2 *For each drawing in Problem 1 that did show a circle packing, draw its nerve.*

Solution:  

Problem 3 *Prove that the nerve of any planar circle packing is a connected, simple, and planar graph.*

Solution: Because a circle packing is connected, given any two circles there is a path between any point on one to any point on the other, so keeping track of tangencies along this path gives a path in the nerve. The nerve is simple because no circle is tangent to itself, and two circles are tangent at most once. Finally, the nerve is planar because it can already be drawn on the plane, and edges don't cross because the circles don't overlap.

Recall that last week we defined the sphere S^2, which we showed that we can view as the plane plus a point that we call ∞.

Problem 4 *Last week we showed that planar graphs can be viewed as graphs on S^2 (without crossing edges) and vice versa. Show that planar circle packings are circle packings on S^2 and vice versa.*

Solution: Given a planar circle packing, adding ∞ makes it an S^2 circle packing. Given an S^2 circle packing, the circles have zero area, so there exists a point not lying on a circle. Placing ∞ at that point and projecting gives a planar circle packing by Problem 4 from last week.

2 The Circle Packing Theorem

The main result that we will study is the somewhat miraculous converse to Problem 3.

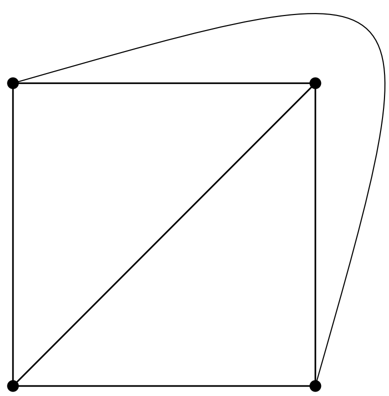
Theorem 1 *(The circle packing theorem) Every connected, simple, planar graph is the nerve of some planar circle packing.*

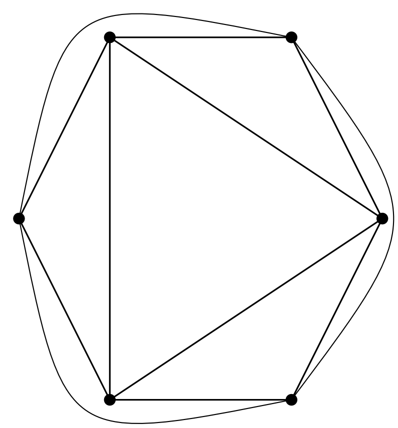
We won't prove this entire theorem this week, but here is one immediate corollary (which can otherwise be tricky to prove).

Problem 5 *Use Theorem 1 to prove Fáry's Theorem, which states that every (connected, simple) planar graph can be drawn in the plane using only straight line segments without crossing edges.*

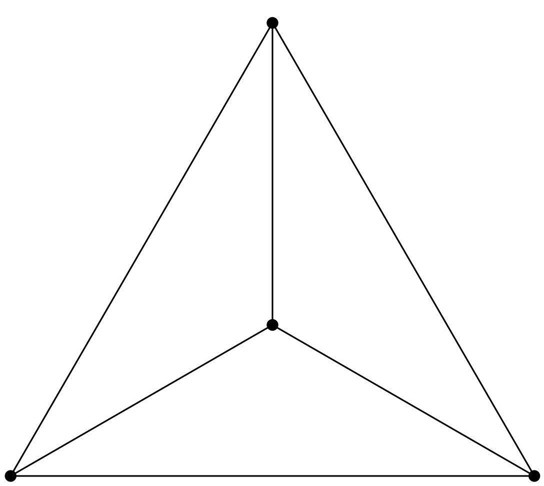
Solution: Given a graph G, by Theorem 1 we find a circle packing whose nerve is G. Connecting the centers of the circle packing with straight lines does not cross edges since the circles don't overlap.

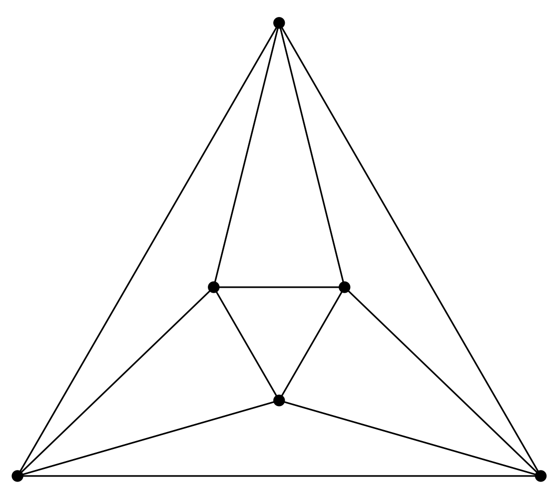
Problem 6 *Last week, we showed that the following two graphs are planar. Redraw them using only straight line segments without crossing edges.*



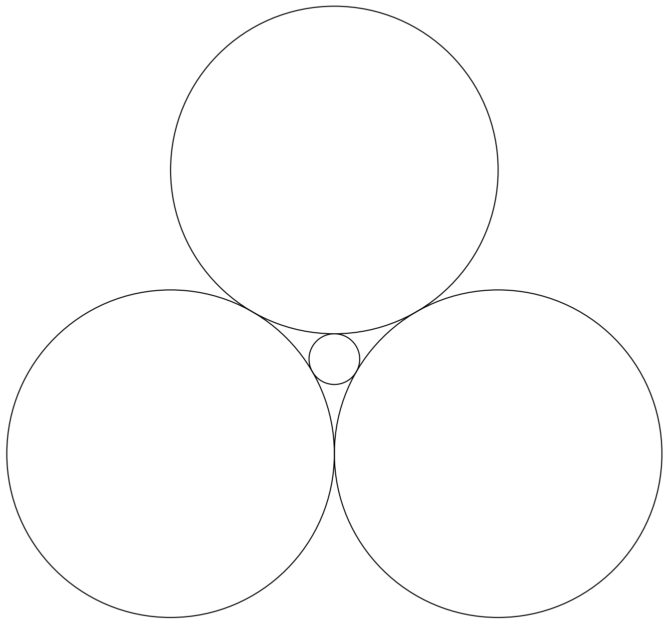


Solution:

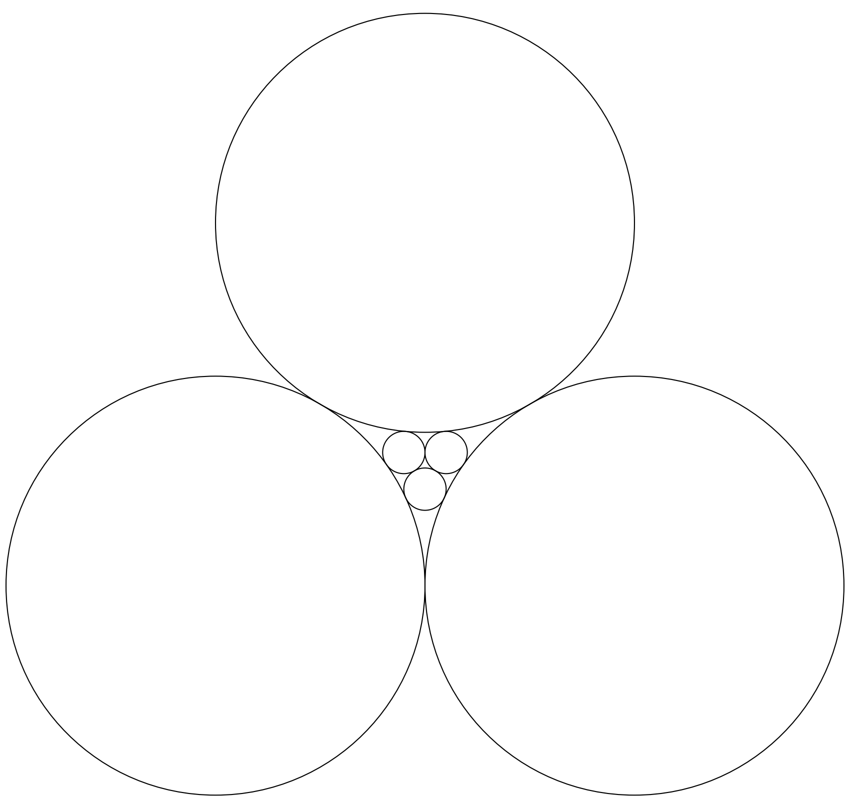




Problem 7 *Find a planar circle packing whose nerve is each planar graph shown on the previous page. (Hint: Using your answer to Problem 6, try putting circles on the picture.*



Solution:



3 Maximal Planar Graphs and Uniqueness

Definition 3 *A planar graph is called **maximal** if no edge can be added to it (without adding vertices) that keeps the graph planar.*

It's a good exercise to check that both of the examples in Problem 6 are actually maximal! We can also check this with the following useful criterion:

Problem 8 *Prove that, for a planar graph G with $V \geq 3$ vertices, the following are equivalent:*

1. *G is a maximal planar graph.*
2. *G is a triangulation - that is, it can be drawn in the plane such that every face (including the exterior face) is a triangle.*
3. *G has exactly $3V - 6$ edges.*

(Hint: To show that three things are equivalent, it's enough to show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. Problem 10 from last week might help for some of those steps.)

Solution: ($1 \Rightarrow 2$) If G were to be drawn with a non-triangular face, we could always split off a triangular face by drawing another edge (which keeps the graph planar), so since G is maximal, this is not possible.

($2 \Rightarrow 3$) Since every edge borders two faces and every face has three edges, $2E = 3F$, and plugging this into the Euler characteristic equation gives $E = 3V - 6$.

($3 \Rightarrow 1$) By Problem 10 from last week, every planar graph has at most $3V - 6$ edges, so if $E = 3V - 6$, then no other edges can be added to G and keep it planar.

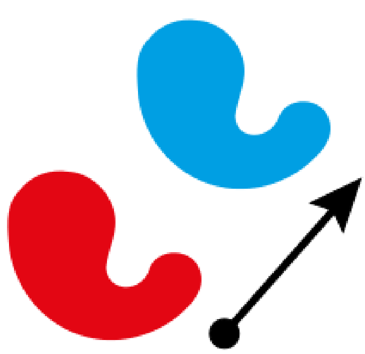
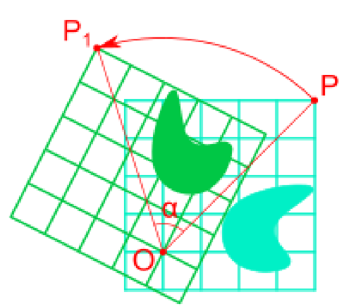
Problem 9 *Show that it suffices to prove Theorem 1 for the case of maximal planar graphs. (Hint: For a non-maximal planar graph, add a "helper" vertex in the middle of every non-triangular face and connect it to every vertex along that face. Why does this make the graph maximal? After turning it into a circle packing, why can you remove the helper circles corresponding to the helper vertices?)*

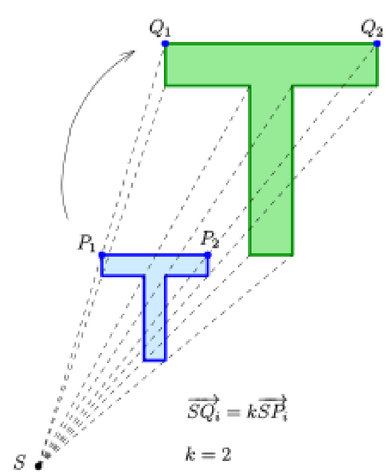
Solution: The helper vertices make the graph planar, because every face is now either a triangle from the original graph, or bounded by edges between the helper vertex and two vertices of the original graph - these are triangles. Since the helper vertices are in the middle of a face, the helper circles will be in the middle of the circles which formed that face originally, and therefore won't be touching any other circles and can be safely removed.

Assuming that graphs are maximal will help prove Theorem 1, because there are fewer maximal planar graphs than planar graphs in general. Maximal planar graphs also give rise to circle packings with a certain "rigidity" property. For this let us define some useful geometric transformations in the plane:

1. A *translation* moves every point in the plane the same amount in a certain direction. A translation of S^2 does this to every point except ∞, and keeps ∞ the same.
2. A *rotation* around some fixed point P moves every other point on the plane as if it were on a circle centered at P. Again, for S^2 this keeps ∞ the same.
3. A *dilation* from some fixed point P stretches or shrinks every other point on the plane as if it were on the ray from P by some constant factor. Again, for S^2 this keeps ∞ the same.

See the below diagrams (courtesy of Wikipedia) for illustrations of a translation, rotation, and dilation, respectively.



The last transformation is best defined on S^2 rather than the plane. For any given circle, define *inversion* about that circle as follows:

- Map the center O of the circle to ∞.
- Map ∞ to O.
- For all other points P, map it to the point P' on the ray OP such that the distances $OP \times OP' = r^2$ where r is the radius of the circle.

Problem 10 *(Bonus) Show that the inversion (about any circle) of a circle is a circle. (Remember that in S^2, lines are circles through ∞.)*

Clearly, translating, rotating, or dilating a circle packing keeps the same nerve (because doing these to any graph keeps it the same graph). You may also want to verify that inverting a circle packing also keeps the same nerve. The following famous theorem states that these are the *only* operations that do so.

Theorem 2 *(Koebe-Andreev-Thurston) For any maximal planar graph, the circle packing given by Theorem* 1 *is unique up to translations, rotations, dilations, and inversions.*

As with many results in graph theory, Theorems 1 and 2 will be most conveniently proved by induction. In Problems 6 and 7, we established the base case (a maximal planar graph with 4 vertices) for Theorem 1. Let us now do so for Theorem 2.

Problem 11 *For the four-vertex graph given in Problem 6, show that Theorem 2 holds. (Hint: Start with any such circle packing, and make an inversion to put ∞ at a spot where two of the circles are tangent. Show that, up to rotation and dilation, there is only one configuration for these two circles. Then show that there is only one way (up to translation) to put the other two circles with the correct tangencies.)*

Solution: Given any such circle packing, by some inversion we place ∞ at one of the tangencies. After stereographic projection, the two circles that were tangent there must become two parallel lines (see Problem 4 from last week), and up to rotation and dilation there is only one pair of parallel lines (two horizontal lines distance 1 apart). Since the last two circles must be tangent to both lines, they must both have radius $1/2$, and since they must be tangent to each other, this leaves only one possible configuration (up to translation).

Problem 12 *Show that Theorem* 2 *holds for the other graph given in Problem* 6.

4 Bonus Section: More application of circle inversions

The fact that inversion takes circles to circles has many useful applications in geometry (in fact, it's the key to solving quite a number of competition problems). In general, it is difficult to tell when (and where) to invert a diagram, but the following heuristics are useful:

- If there are many circles (remember, lines are circles in S^2) which need to intersect at a single point A, invert about a circle centered at A.
- If there are many angles AXB with fixed points A and B, invert about a circle centered at either A or B.
- If there are points which need to lie on the same circle, invert about a circle centered at a point on that circle - it is usually easier to prove that points are collinear than cocyclic.

The following problems illustrate the utility of this technique.

Problem 13 *(Ptolemy's Theorem) For any points A, B, C, and D, $AB \cdot CD + BC \cdot AD \geq AC \cdot BD$, with equality if and only if A, B, C, and D lie on a line or circle in that order.*

Problem 14 *(Feuerbach's Theorem) Given any triangle, show that the feet of the altitudes and the midpoints of the sides lie on the same circle. Furthermore, show that this circle is tangent to the incircle and all three excircles of the triangle.*

Problem 15 *(IMO 1996) Let P be a point inside triangle ABC such that angle APB minus angle ACB equals angle APC minus angle ABC. Let D and E be the incenters of triangles APB and APC, respectively. Show that AP, BD, and CE all intersect at one point.*