Cardinality

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1 Review of Functions

Today we will discuss sizes, or cardinality, of different sets. In order to do this, we must recall certain properties of functions.

Definition 1 *Let $f : X \to Y$. X is the* ***domain*** *of f, and Y is the* ***codomain*** *of f. f is called* ***injective*** *if $f(x) = f(y)$, then $x = y$. (f sends different inputs to different outputs). f is called* ***surjective*** *if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$ (every element of Y comes from an element in X). f is called* ***bijective*** *it is injective and surjective.*

Problem 1 *Please draw what it means for a function to be injective, surjective, bijective, and none of these?. Can you give explicit examples of strictly injective, surjective, and bijective function? (i.e. only injective, only surjective, bijective).*

Problem 2 *Can you come up with strictly injective, surjective, and bijective functions with infinite domain and codomain?*

Problem 3 *For each $n \in \mathbb{N}_{>0}$, let the set $[n] = \{1, \dots, n\}$. Let X be a finite set. Prove that there exists a unique $n \in \mathbb{N}_{>0}$ such that there is a bijection between X and $[n]$. (For uniqueness: suppose $m \neq n$. If X is in bijection with $[m]$ and $[n]$, derive a contradiction).*

Problem 4 *(Cantor's Theorem) Let X any set. Let $\mathcal{P}(X)$ be the set of all subsets of X. Let $f : X \to \mathcal{P}(X)$ be any function. Show that f is not a surjection. (Hint: Consider the set $\{x \in X : x \notin f(x)\}$. This is a subset of X. Can something map to this set?)*

2 Cardinality

Consider this thought experiment called Hilbert's Hotel. Imagine a hotel with an infinite number of rooms, labeled $1, 2, 3, \ldots$. Suppose there is an extremely hard-working front desk employee who will always accommodate guests when possible.

A man walks to the desk and rings the bell, asking for a room. The employee made room for him by asking the guests in room 1 to move to room 2, the guests in room 2 to move to room 3, and so on. Then, the man can occupy room 1, and every original guest still has a room. The same process can be done for any finite number of guests.

Suppose an infinitely long bus pulls up to the hotel with countably many guests in the bus asking for rooms. The employee was stunned, until he realized he could do the following: guests in room 1 were moved to room 2, guests in room 2 were moved to room 4, guests in room 3 were moved to room 6, and so on. Thus, there were enough rooms for the new incoming guests.

One night, a countably infinitely long line of countably infinite long busses pulls up. The employee is stressed, as he would miss out on an infinite amount of money. However, he remembered that there are an infinite number of prime numbers. Using this, he asks guests in room 1 to move to room 2^1, guests in room 2 to move to room 2^2, guests in room 3 to move to room 2^3 and so on. Then, the first bus will get rooms of powers of the next prime, 3. He can do this an infinite number of times, so everyone is happy.

In all of this, the employee managed to fit an infinite number of infinite busses into a full infinite hotel.

Definition 2 *Let X, Y be sets. We say that X and Y have the same cardinality, denoted by $|X| = |Y|$ if there exists a bijection $f : X \to Y$. We write $|X| \leq |Y|$ if there exists an injection $f : X \to Y$. We write $|X| \geq |Y|$ if there exists a surjection $f : X \to Y$. (We can make these inequality symbols strict if we know there is no bijection).*

Problem 5 *Let X be a set, y an element such that $y \notin X$. Let $X \times \{y\} = \{(x, y) : x \in X\}$. Show that $|X| = |X \times \{y\}|$*

Problem 6 *Let X_1, X_2, Y_1, Y_2 be sets such that $|X_1| = |Y_1|$ and $|X_2| = |Y_2|$. Show that $|X_1 \cup X_2| = |Y_1 \cup Y_2|$ and $|X_1 \times X_2| = |Y_1 \times Y_2|$ where $A \times B = \{(a, b) : a \in A, b \in B\}$.*

Problem 7 *In "The Fault in Our Stars" there is a quote: "There are infinite numbers between 0 and 1. There's .1 and .12 and .112 and an infinite collection of others. Of course, there is a bigger infinite set of numbers between 0 and 2, or between 0 and a million". Show that this is not true, i.e. show that* $|[0,1]| = |[0,2]| = |[0,1000000]|$.

Definition 3 *A set* X *is called countable if* $|X| = |\mathbb{N}|$. *Otherwise, if* X *is infinite it is called uncountable. If* X *is countable, we can enumerate* X *by* $x_1, x_2, x_3, \ldots$.

Problem 8 *The next sentence in the previous quote is: "Some infinities are bigger than other infinities". Show that* $(0,1)$ *is uncountable. (Clearly* $(0,1)$ *is infinite. Suppose that* $(0,1)$ *is countable. Then you can enumerate* $(0,1)$*, say by*
$x_1 = 0.a_1^1 a_2^1 a_3^1 \ldots$
$x_2 = 0.a_1^2 a_2^2 a_3^2 \ldots$
$x_3 = 0.a_1^3 a_2^3 a_3^3 \ldots$
Can we find a number which is not in this list?)

Problem 9 *Show that $\mathbb{Z}$ is countable. (Explicitly define such a function)*

Problem 10 *Let X, Y be countable sets. Show that $X \times Y = \{(x, y) : x \in X, y \in Y\}$ is countable. Hint: Try drawing a visual representation of $X \times Y$. For example, $[n_1] \times [n_2]$ can be thought of as*

$$\begin{array}{cccc} (1,1) & (1,2) & \dots & (1,n_2) \\ (2,1) & (2,2) & \dots & (2,n_2) \\ \vdots & \vdots & \ddots & \vdots \\ (n_1,1) & (n_1,2) & \dots & (n_1,n_2) \end{array}$$

Problem 11 *Using problems 9 and 10, show that $\mathbb{Q}$ is countable. Can we think of $\mathbb{Q}$ as the cross product of two countable sets? For example, what would the fraction $\frac{p}{q}$ look like?*

Problem 12 *Using problems 8, 11, show that the irrationals $\mathbb{R}\backslash\mathbb{Q}$ are uncountable.*

Problem 13 *Visually draw a bijection from the unit circle to the real line $\mathbb{R}$. (Hint: Remember the stereographic projection from last week.)*

Problem 14 *Explicitly show, without any of the previous problems, that* $\mathbb{N} \times \mathbb{N}$ *is countable using a different method than the one described in the thought experiment at the beginning of the section.*

Problem 15 *Let* $f, g : \mathbb{N} \to \mathbb{N}$*. We say that* f *dominates* g *if* $f(x) \geq g(x)$ *for all* $x \in \mathbb{N}$*. Let* X *be a set of functions such that for any* $g : \mathbb{N} \to \mathbb{N}$ *there exists a function* $f \in X$ *such that* f *dominates* g*. Show that* X *is uncountable. (Hint: Suppose* X *is countable. Enumerate* X *by* $f_1, f_2, \ldots$*. Construct a function which is not dominated by any function* f_i*)*

3 Bonus: Ordinals and Cardinals

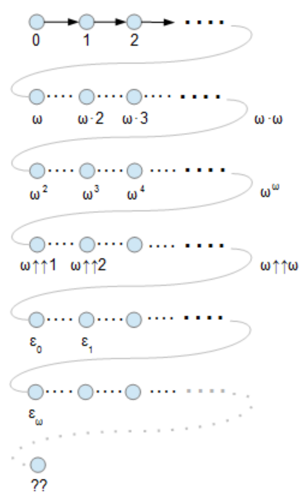
Definition 4 *A set X is called well-ordered if any nonempty subset of X has a least element.*

To motivate ordinals, we always want to have a notion of "the next number". For finite numbers, we clearly have the natural numbers. Ordinals help us generalize this concept to infinite scales. Defining ordinals precisely is a bit out of the scale of our level. However, we can have a good understanding of the structure.

The finite ordinal numbers are $0, 1, 2, 3, \ldots$ all of the natural numbers. We want these to have a nice property, specifically the well-ordering property. To make sense of numbers containing other numbers, we construct them inductively as sets:

$$0 = \emptyset$$
$$1 = \{0\}$$
$$2 = \{0, 1\}$$
$$\vdots$$

The first infinite ordinal is $\omega = \{0, 1, 2, \ldots\}$. Then, we have $\omega + 1 = \omega \cup \{\omega\}$. We can continue this infinitely many times. We can imagine $\omega + 1$ as a copy of the natural number line except with an element all the way to the right.



Problem 16 *Write out explicitly the ordinals $0, 1, 2, 3, 4$ without using ordinal numbers (i.e. only use the symbols $\{, \}, \emptyset$).*

Definition 5 *A cardinal is an ordinal which is not in bijection with any smaller ordinal. Formally, the cardinality of a set X is the least ordinal number α such that $|X| = |\alpha|$.*

Definition 6 *The successor cardinal of cardinal κ denoted κ^+ is the least cardinal greater than κ. A limit cardinal is a cardinal κ such that for every cardinal $\lambda < \kappa$, we have that $\lambda^+ < \kappa$.*

Problem 17 *Give two examples of infinite cardinals, one which is a limit cardinal and one which is a successor cardinal.*

Problem 18 *Think about the differences between cardinals and ordinals. Give an example of two distinct ordinals which have the same cardinality. What would arithmetic on the ordinals look like? What about the cardinals?*