Logic II

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1   Logical Equivalence

Last time, we filled out this truth table. However, there are two columns that have the same truth values: $A \Rightarrow B$ and $\neg A \vee B$. (Also A and $(A \Rightarrow B) \Rightarrow A$ but the former is more important)

A	B	$\neg A$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$A \Leftrightarrow B$	$A \vee \neg A$	$\neg A \vee B$	$(A \Rightarrow B) \Rightarrow A$
T	T	F	T	T	T	T	T	T	T
F	T	T	T	F	T	F	T	T	F
T	F	F	T	F	F	F	T	F	T
F	F	T	F	F	T	T	T	T	F

There are many statements which are logically equivalent, and it always suffices to check with a truth table. There are a few tricks to help us see logical equivalences much easier.

Theorem 1. *(De Morgan's Laws) For statements A, B we have that*
1. $\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$ i.e. $\neg(A \vee B)$ is Logically Equivalent to $(\neg A) \wedge (\neg B)$
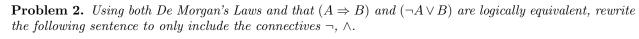*2. $\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$ i.e. $\neg(A \wedge B)$ is Logically Equivalent to $(\neg A) \vee (\neg B)$*

Before moving on, check to make sure you understand why the theorem is true!

Problem 1. *Using De Morgan's Laws, rewrite the following sentences to only include the connectives $\neg, \wedge$.*

$(A \vee B)$

$((A \vee \neg B) \vee C)$

$(A \wedge B) \vee (\neg C \wedge D)$

Problem 2. *Using both De Morgan's Laws and that $(A \Rightarrow B)$ and $(\neg A \vee B)$ are logically equivalent, rewrite the following sentence to only include the connectives $\neg$, $\wedge$.*

$(A \Rightarrow B)$

$((A \Rightarrow \neg B) \vee C)$

$((A \Rightarrow B) \vee (\neg C \vee D))$

*Using your observations from above, which of the following symbols do we not strictly need: $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$?
What about our quantifiers $\exists$, $\forall$?*

Here are a few more simple useful identities. Make sure they make sense to you!

Theorem 2 (Useful properties)**.**

  1. $\neg\neg A \Leftrightarrow A$.

  2. $A \vee T \Leftrightarrow T$ and $A \vee F \Leftrightarrow A$.

  3. $A \wedge T \Leftrightarrow A$ and $A \wedge F \Leftrightarrow F$.

  4. $A \vee \neg A = T$ and $A \wedge \neg A = F$.

  5. *(distributive law)* $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$ and $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$.

Problem 3. *Using the above properties, De Morgan's Laws, and/or $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$, simplify the following expressions:*

  1. $((\neg A \Rightarrow (\neg B)) \Rightarrow (A \wedge B))$

  2. $(((A \vee B) \Rightarrow (\neg C \wedge \neg B)) \wedge (\neg B \Rightarrow C))$

(By the way, these were expressions from last week's packet for which you were asked to find an assignment that makes them true. It's much easier after simplifying!)

2 Models

Recall that a **Propositional Language** is a nonempty set of **Atomic Propositions** ex. $\mathcal{L} = \{A_1, A_2, A_3, A_4\}$ where $\mathcal{L}$ is the (propositional) language, and A_i are the atomic propositions.

Definition 1. *Let $\mathcal{L}$ be a language. An $\mathcal{L}$-Model $\mathcal{M}$ is a subset of $\mathcal{L}$. You can think of $\mathcal{M}$ as the "true" statements in $\mathcal{L}$.*

Definition 2. *Let $\mathcal{L}$ be a language, φ an $\mathcal{L}$-sentence, $\mathcal{M}$ an $\mathcal{L}$-Model. We say "$\mathcal{M}$ models φ" (intuitively, "φ is true in $\mathcal{M}$"), denoted by the relation $\mathcal{M} \models \varphi$ defined as follows:*

1. *If φ is an atomic proposition, and $\varphi \in \mathcal{M}$ then $\mathcal{M} \models \varphi$*

2. *If φ is of the form $(\psi \wedge \theta)$ then $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi$ and $\mathcal{M} \models \theta$*

3. *If φ is of the form $(\psi \vee \theta)$ then $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi$ or $\mathcal{M} \models \theta$*

4. *If φ is of the form $(\psi \Rightarrow \theta)$ then $\mathcal{M} \models \varphi$ iff*

   (a) *$\mathcal{M} \models \psi$ and $\mathcal{M} \models \theta$ OR*

   (b) *$\mathcal{M} \not\models \psi$ i.e. $\mathcal{M}$ does not model ψ*

5. *If φ is of the form $(\neg\psi)$ then $\mathcal{M} \models \varphi$ if $\mathcal{M} \not\models \psi$*

Problem 4. *Suppose our language is given by $\mathcal{L} = \{All\ weather\ conditions\} \cup \{All\ physical\ conditions\}$ the set of all weather conditions (rain, snow, sunny, cloudy, etc.) and all physical conditions (hot, cold, sick, tired, etc.) Let $\mathcal{M} = \{It\ is\ hailing, It\ is\ cloudy, I\ am\ sick\}$. What sentences are modeled by considering the conditions in $\mathcal{M}$ to be true?*

It is not cloudy

It is hailing or I am tired

(If it is raining, then I am cold) and (If it is not sunny, then I am hot or tired)

Problem 5. *Let $\mathcal{L} = \{A_1, A_2, A_3, A_4, A_5\}$ be a language. Consider the $\mathcal{L}$-Model $\mathcal{M} = \{A_2, A_4\}$. Decide, using definition 2, if $\mathcal{M}$ does or does not model the following sentences:*
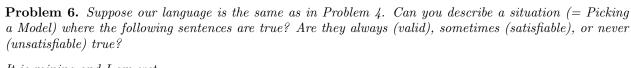
A_3

$(\neg A_1)$

$(A_2 \vee A_5)$

$((A_3 \vee A_4) \Rightarrow (A_1 \wedge A_2))$

$(A_1 \vee (\neg A_1))$

Definition 3. *Let $\mathcal{L}$ be a language, φ an $\mathcal{L}$-sentence. We say that:*

1. *φ is **valid** if for all $\mathcal{L}$-Model $\mathcal{M}$, we have $\mathcal{M} \models \varphi$*

2. *φ is **satisfiable** if there exists an $\mathcal{L}$-Model $\mathcal{M}$ such that $\mathcal{M} \models \varphi$*

3. *φ is **unsatisfiable** if for all $\mathcal{L}$-Model $\mathcal{M}$, we have $\mathcal{M} \not\models \varphi$*

Recall that you can think of models as choosing which atomic propositions are "true". In the challenge section, we will define this more precisely. However, this means that to decide if a sentence is valid, consistent, or inconsistent, it is enough to use a truth table.

Problem 6. *Suppose our language is the same as in Problem 4. Can you describe a situation (= Picking a Model) where the following sentences are true? Are they always (valid), sometimes (satisfiable), or never (unsatisfiable) true?*

It is raining and I am wet

(If it is sunny, then I am hot and tired) and (I am not tired)

(It is snowing) and (If it is snowing, then I am cold) and (I am not cold)

Problem 7. *Let $\mathcal{L} = \{A_1, A_2\}$ be a language. Decide whether the following sentences are valid, satisfiable, or unsatisfiable. If the sentence is satisfiable, give a model $\mathcal{M}$ which models the sentence.*

$((\neg(A_1 \Rightarrow A_2)) \Rightarrow A_1)$

$((\neg(A_1 \Rightarrow A_2)) \Rightarrow A_2)$

$((A_1 \Rightarrow (\neg A_2)) \Rightarrow (A_1 \wedge A_2))$

3   Deductions and Proofs

Definition 4. *Let $\mathcal{L}$ be a language. An $\mathcal{L}$-theory is a collection of $\mathcal{L}$-sentences.*

You can think of a theory as the set of sentences which are considered true, similar to how models pick which atomic propositions are true.

Definition 5. *Let $\mathcal{L}$ be a language, Σ an $\mathcal{L}$-theory with $\varphi, (\varphi \Rightarrow \psi) \in \Sigma$. The Rule of Inference i.e. Modus Ponens says that we can infer ψ.*

Definition 6. *Let $\mathcal{L}$ be a language, Σ and $\mathcal{L}$-theory. Let φ be a sentence. Then we write "$\Sigma \vdash \varphi$" or "Σ proves φ" if there exists a finite sequence of $\mathcal{L}$-sentences $\theta_1, \theta_2, \ldots, \theta_n$ such that the following hold:*

1. *$\theta_n = \varphi$*

2. *For each $m \leq n$ either:*

    (a) *θ_m is valid*

    (b) *$\theta_m \in \Sigma$*

    (c) *θ_m is inferred from 2 previous sentences in the sequence by Modus Ponens i.e. if $\theta_k = \psi$ and $\theta_{k+1} = (\psi \Rightarrow \sigma)$, then we can write $\theta_{k+2} = \sigma$*

You can think of the above as formalizing a "proof structure" using strictly mathematical language. For realistic purposes, we often include English words or phrases which help connect different steps of our proof. For example, we may write "by the Cauchy-Schwarz inequality, it follows that ...". However, here we do not need to write any such phrases. We only need to make sure the sequence $\theta_1, \theta_2, \ldots, \theta_n$ follows the above rules.

Problem 8. *Suppose $\mathcal{L} = \{All\ mathematical\ statements\}$. Consider the $\mathcal{L}$-theory $\Sigma = \{Triangle\ Inequality\ and\ Young's\ Inequality,\ Triangle\ Inequality \Rightarrow Cauchy\text{-}Schwarz\ Inequality,\ Stone\text{-}Weierstrass\ Theorem\ or\ Young's\ Inequality\}$ i.e. Σ tells us which statements we will treat as true statements. Can we prove the Cauchy-Schwarz Inequality using the rules in Definition 6 and what we have in Σ? (Hint: if we have $A \wedge B$, what sentence can we write which is always true?)*

Problem 9. *Let $A, B, C \in \mathcal{L}$. Suppose that $\{(\neg A \lor B), (B \Rightarrow C), A\} \subseteq \Sigma$ where Σ is an $\mathcal{L}$-theory. Prove that $\Sigma \vdash C$. (Hint: You may need to find a valid sentence using A, B. Think about what you are given, and what you can get from Modus Ponens)*

4   Logic Puzzles

Problem 10. *You are in a deep dream, and encounter a room with two doors, and two hooded figures. One stands in front of door A, and the other in front of door B. As you approach the doors, a sign appears: "One door leads to riches, the other to famine. One will tell you lies, the other truths. Ask only one of them exactly one question, then select a door." What question should you ask? Can you always choose riches?*

Problem 11. *You are traveling through the deep ocean, and stumble upon an island. Upon closer inspection, you see 100 red-eyed dragons on the island. However, they are all enclosed behind barbed wire. There is a king who sits in his tower observing the captured red dragons. As you approach, he invites you to talk to him. He tells you that the dragons cannot exchange information about each other's eye color, and there are no reflections. Hence, none of the dragons know their own eye color. Every day at nightfall, a guard stands by the gate. The dragons are allowed to guess what their own eye color is. If they are correct, they are set free. If they are wrong, they will be executed. Amused, the king asks you to go to the speaker and say one sentence. You think hard, and exclaim: "At least one of you has red eyes!" Thinking nothing of it, the king lets you leave. What will happen on nights 1-100?*

5   Challenge Problems

Definition 7. *Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-model, and Σ an $\mathcal{L}$-theory. Then, we write $\mathcal{M} \models \Sigma$ if for all $\varphi \in \Sigma$ we have that $\mathcal{M} \models \varphi$.*

Problem 12. *Let $\mathcal{L} = \{A_i : i \in \mathbb{N}\}$. Let $\Sigma = \{A_i \Rightarrow A_i + 2 : i \in \mathbb{N}\} \cup \{A_1\}$.*

1. *Find $\mathcal{M}_1, \mathcal{M}_2$ $\mathcal{L}$-models such that $\mathcal{M}_1 \neq \mathcal{M}_2$ and both $\mathcal{M}_1 \models \Sigma$ and $\mathcal{M}_2 \models \Sigma$. Justify your answer.*

2. *Prove that Σ has no finite models.*

Problem 13. *Find a counterexample to the following: If $\mathcal{M}_1 \models (\varphi \Rightarrow \psi)$ and $\mathcal{M}_2 \models (\varphi \Rightarrow \psi)$ then $\mathcal{M}_1 \cap \mathcal{M}_2 \models (\varphi \Rightarrow \psi)$*

Definition 8. *Let $\mathcal{L}$ be a language, Σ and $\mathcal{L}$-theory. Then Σ is satisfiable if there exists an $\mathcal{L}$-model $\mathcal{M}$ such that $\mathcal{M} \models \Sigma$.*

Problem 14. *Let $\mathcal{L} = \{A_i : i \in \mathbb{N}\}$. For each of the following conditions, give an example of a non-empty $\mathcal{L}$-theory Σ which is **not satisfiable** and meets the appropriate condition.*

1. *Each member of Σ is satisfiable by itself*

2. *For any two members φ_1, φ_2 of Σ, the theory $\{\varphi_1, \varphi_2\}$ is satisfiable*

3. *For any three members $\varphi_1, \varphi_2, \varphi_3$ of Σ, the theory $\{\varphi_1, \varphi_2, \varphi_3\}$ is satisfiable*

4. *For any finite collection of members $\varphi_1, \ldots, \varphi_n$ of Σ, the theory $\{\varphi_1, \ldots, \varphi_n\}$ is satisfiable*