ORMC AMC Group: Week 7
Polynomials

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1 Basic Properties

A polynomial in one variable is an expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 x_0.$$

The values a_i are called the coefficients. The degree of the polynomial, $\deg(p(x)) = n$, is the highest power of x such that x^n has a non-zero coefficient. Polynomials with lower degree also have special names: degree 1 are linear polynomials, degree 2 are quadratic, degree 3 are cubic, and so on.

1.1 Arithmetic of Polynomials

Let p, q be two polynomials where

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 x_0,$$

$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0 x_0.$$

Then polynomial addition/subtraction is done by adding/subtracting the coefficients for each term:

1. $p(x) + q(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_0 + b_0)x_0$
2. $p(x) - q(x) = (a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \cdots + (a_0 - b_0)x_0$

To multiply two polynomials, multiply every pair of terms and take their sum. For example,

$$(a_1 x + a_0)(b_1 x + b_0) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0.$$

In general, the i-th coefficient in $p(x)q(x)$ is

$$a_i b_0 + a_{i-1} b_1 + \cdots + a_1 b_{i-1} + a_i b_0.$$

1.2 Exercises

1. If f is a quadratic polynomial such that $f(-1) = 1, f(0) = 2, f(2) = 3$, find f.

2. Given $\deg(p(x)) = a, \deg(q(x)) = b$, what can we say about the degrees of:

   (a) $p(x) + q(x)$

   (b) $p(x)q(x)$

   (c) $p(q(x))$

1.3 Problems

1. What is the sum of the coefficients in the expansion $(x + 2y - 1)^6$?

2. Let $Q(x) = x^2 + 2x + 3$, and suppose that $P(x)$ is a polynomial such that $P(Q(x)) = x^6 + 6x^5 + 18x^4 + 32x^3 + 35x^2 + 22x + 8$. Compute $P(2)$.

3. Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b, b + c$, and $c + a$. Find t.

4. (Hard) Let $a_1, a_2, \ldots, a_{2022}$ be real numbers such that

$$a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \cdots + a_{2022} \cdot 2022 = 0$$
$$a_1 \cdot 1^2 + a_2 \cdot 2^2 + a_3 \cdot 3^2 + \cdots + a_{2022} \cdot 2022^2 = 0$$
$$\vdots$$
$$a_1 \cdot 1^{2021} + a_2 \cdot 2^{2021} + a_3 \cdot 3^{2021} + \cdots + a_{2022} \cdot 2022^{2021} = 0$$

and

$$a_1 \cdot 1^{2022} + a_2 \cdot 2^{2022} + a_3 \cdot 3^{2022} + \cdots + a_{2022} \cdot 2022^{2022} = 1.$$

What is the value of a_1?

2 Roots of Polynomials

Given a polynomial, $p(x)$, a root is a value, r, such that $p(r) = 0$. The fundamental theorem of algebra states that every degree n polynomial has exactly n complex roots, $r_1, r_2, \ldots, r_n$.

2.1 Unique Factorization of Polynomials

Any polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x_0$ can be expressed in the form

$$p(x) = a_n(x - r_1)(x - r_2)\ldots(x - r_n)$$

where $r_1, r_2, \ldots, r_n$ are the roots of $p(x)$.

2.2 Remainder Theorem

1. (Remainder Theorem) $f(k)$ is the remainder when $f(x)$ is divided by $x - k$.
2. (Factor Theorem) $x - r$ is a factor of $f(x)$ if and only if $f(r) = 0$.

2.3 Exercises

1. What is the remainder when $x^{51} + 51$ is divided by $x + 1$?
2. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has roots $1, 3, 5, 7$. Determine all the coefficients of $f(x)$.

2.4 Problems

1. For what value(s) of k is $x - 2$ a factor of $x^3 + 2kx^2 + k^2x + k - 4$?
2. There is a unique polynomial $P(x)$ of the form

   $$P(x) = 7x^7 + c_1x^6 + c_2x^5 + \cdots + c_6x + c^7$$

   such that $P(1) = 1, P(2) = 2, \ldots,$ and $P(7) = 7$. Find $P(0)$.
3. The polynomial $g(x) = x^3 - x^2 - (m^2 + m)x + 2m^2 + 4m + 2$ is divisible by $x - 4$ and all of the roots of $g(x)$ are integers. Find m.
4. Find the remainder when $x^{28} + 1$ is divided by $x^4 + x^3 + x^2 + x + 1$.

3 Vieta's Formulas

Vieta's formulas can be derived by expanding the previous equality:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x_0 = c(x - r_1)(x - r_2)\ldots(x - r_n)$$

where $c = a_n$. This gives us the following identities:

1. Sum of roots: $r_1 + r_2 + \cdots + r_{n-1} + r_n = -\dfrac{a_{n-1}}{a_n}$
2. Product of roots: $r_1 r_2 \ldots r_n = (-1)^n \dfrac{a_0}{a_n}$

In general, Vieta's formulas can also be written equivalently as

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} r_{i_j} \right) = (-1)^k \frac{a_{n-k}}{a_n}.$$

3.1 Exercises

1. Find the sum of the roots of the polynomial $3x^4 - 5x^2 + 2x + 3$.

2. Let r, s, and t be the roots of the polynomial $x^3 + 2x^2 + 3x + 4$. Find $r^2 + s^2 + t^2$.

3. Find the sum of the reciprocals of the roots of the polynomial $10x^2 + 5x + 1$.

3.2 Problems

1. For how many positive integers a does the polynomial

   $$x^2 - ax + x$$

   have an integer root?

2. Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11, P(11) = 17, P(17) = 23$.

3. Let r, s, and t be the three roots of the equation

   $$8x^3 + 1001x + 2008 = 0.$$

   Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

4. (Hard) Let $r_1, \ldots, r_n$ be the distinct real zeroes of the equation

   $$x^8 - 14x^4 - 8x^3 - x^2 + 1 = 0.$$

   Evaluate $r_1^2 + \cdots + r_n^2$.

4 Binomial Theorem

Consider the algebraic expansion of the term $(x+y)^n$. The first few terms are:

1. $(x+y)^2 = x^2 + 2xy + y^2$
2. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
3. $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Notice that the coefficients in the expansion of $(x+y)^n$ form the n-th row of Pascal's triangle. Consider the product of n copies of $(x+y)$:

$$(x+y)^n = \underbrace{(x+y)(x+y)\dots(x+y)}_{n \text{ copies}}$$

The coefficient of $x^k y^{n-k}$ is equivalent to choosing k of the n terms from which to pick x. By definition, there are $\binom{n}{k}$ such subsets. Thus, the binomial theorem states that

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.\end{aligned}$$

4.1 Examples

1. Expand the expression $(1-x)^5$.

2. Find the coefficient of x^8 in the expansion $(x+5)^{10}$.

3. Evaluate $1001^3 - 3 \cdot 1001^2 + 3 \cdot 1001 - 1$.

4.2 Problems

1. Find the value of

$$\binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6.$$

2. Find the remainder when 65^{20} is divided by 512.

3. What is the hundreds digit of 2011^{2011}?

4. Find the units digit of the expansion of

$$(15+\sqrt{220})^{19}+(15+\sqrt{220})^{82}.$$

5. (Hard) Evaluate

$$\sum_{n=1}^{\infty}\frac{n^5}{n!}.$$