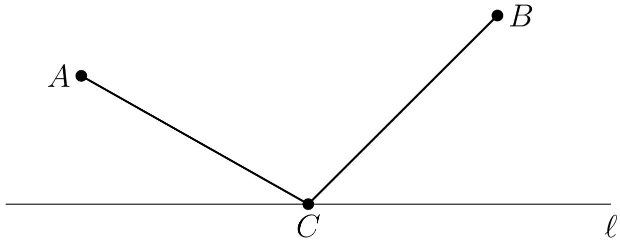
Geometric optima

Nakul and Andreas

Theorem 1 (Triangle inequality)**.**
For any three points A, B, C, $AB + BC \geq AC$. Moreover, if B does not belong to the segment AC, the inequality is strict.

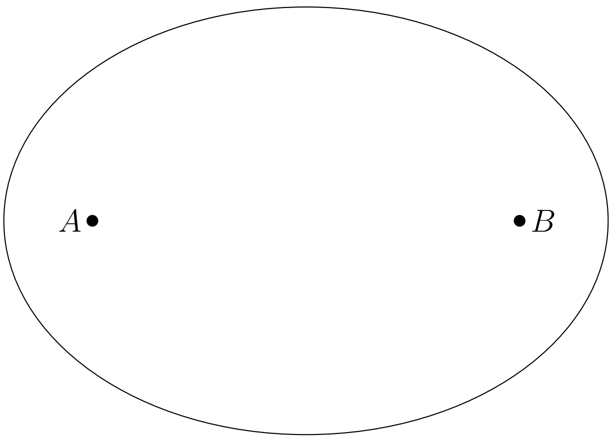
Problem 1.
(a) Let A and B be two points on the same side of some line ℓ. Find point C on ℓ that minimizes $AC + BC$.



[Hint: reflect B on ℓ.]

(b) Assume that ℓ is a mirror. If one shines a laser from A to C, would it pass through B? If yes, why? If not, why not?

(c) Consider an ellipse E, with foci A and B. By definition, E is the set of all points C that satisfy $AC + BC = k$ for some fixed constant $k > 0$. Assuming that the ellipse is a reflective surface, find the point(s) on the ellipse that A can aim a laser at so that it reaches B.



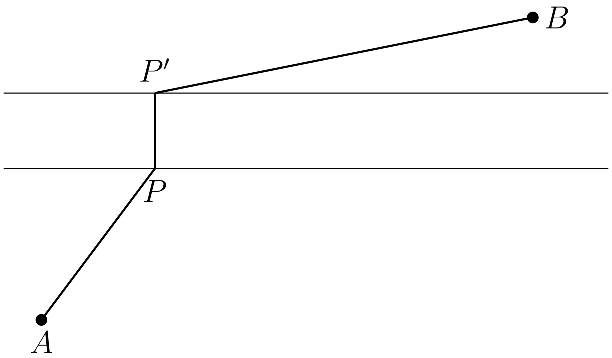
[Hint: fix a point P on the ellipse, draw a tangent line at that point and use the previous parts.]

Problem 2.
In a convex quadrilateral $ABCD$, find the point T for which the sum of the distances to the vertices is minimal.

Problem 3.
Given an angle and two points C and D in its interior, find points A and B on the sides of the angle such that $CA + AB + BD$ is a minimum.

Problem 4.
Let C be a given point in the interior of a given angle. Find points A and B on the sides of the angle such that the perimeter of the triangle ABC is a minimum.

Problem 5.
A road needs to be constructed from town A to town B, crossing a river, over which a perpendicular bridge is to be constructed.

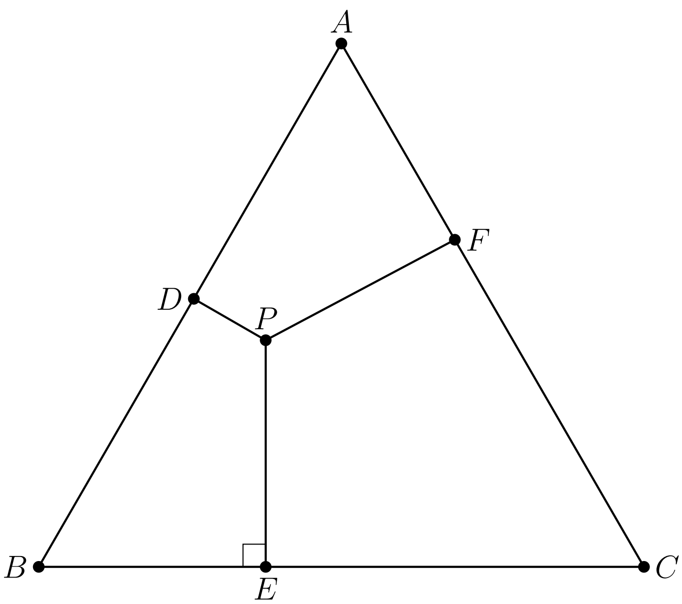


Where should the bridge be placed to minimize $AP + PP' + P'B$?

Problem 6.
Prove that the rectangle of maximal area inscribed in a circle is a square.

Problem 7.
Given an angle and a point P in its interior, construct a line through the given point that cuts off from the angle a triangle of minimal area.

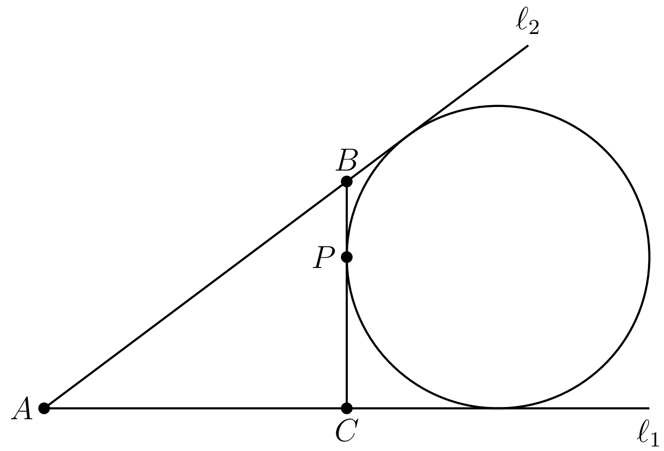
Problem 8.
ΔABC is equilateral. Let P be a point in the interior. PD, PE, and PF are the perpendiculars from P to the sides of the triangle.



(a) Find the point(s) P such that $PD + PE + PF$ is minimized.

(b) Find the point(s) Q such that $QA + QB + QC$ is minimized. What if the triangle is not equilateral?

Problem 9.
Consider two intersecting tangent lines ℓ_1 and ℓ_2 to a circle. Denote their point of intersection A.



(a) Let P be a point on the circle between the tangents, and BC be the tangent at the point P. Describe how P should be selected in order to minimize the perimeter of ΔABC.

(b) Now assume that ℓ_1 and ℓ_2 are intersecting lines and P is a fixed point in the interior of the angle. Pass a line BC going through P that cuts off from the angle, a triangle ΔABC of minimal perimeter.

Problem 10.
For this problem, we will use the phrase "maximal n-gon" to denote the n-gon with maximal area among other n-gons of equal perimeter.
(a) Show that a maximal 3-gon is an equilateral triangle.

(b) Show that a maximal 4-gon is a rhombus.

(c) More generally, show that a maximal n-gon is regular i.e. it has equal sides and equal angles.