THE 1-9 GAME

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INTERMEDIATE 2

1. The 1-9 Game

Let's play a game. I have a special deck of cards that has a card numbered 1, 2, . . . , 9 (so there are exactly 9 cards). I will lay out the cards *face up* on the table. Then we will start the game. Here's how you play: on your turn, you can pick up any unselected card and put it into your hand. If at any point in the game, your hand has exactly 3 cards that sum up to 15, you win the game. Otherwise, we keep playing until somebody wins or all the cards are selected. Here's an example of how the game could go:

(1) I select the number 9
(2) You select the number 1
(3) I select the number 6 (notice that $9 + 6 = 15$, but the game is *not* over, because I need to use 3 cards to win)
(4) You select the number 4
(5) I select the number 2
(6) You select the number 3
(7) I select the number 7 and win, because $2 + 6 + 7 = 15$

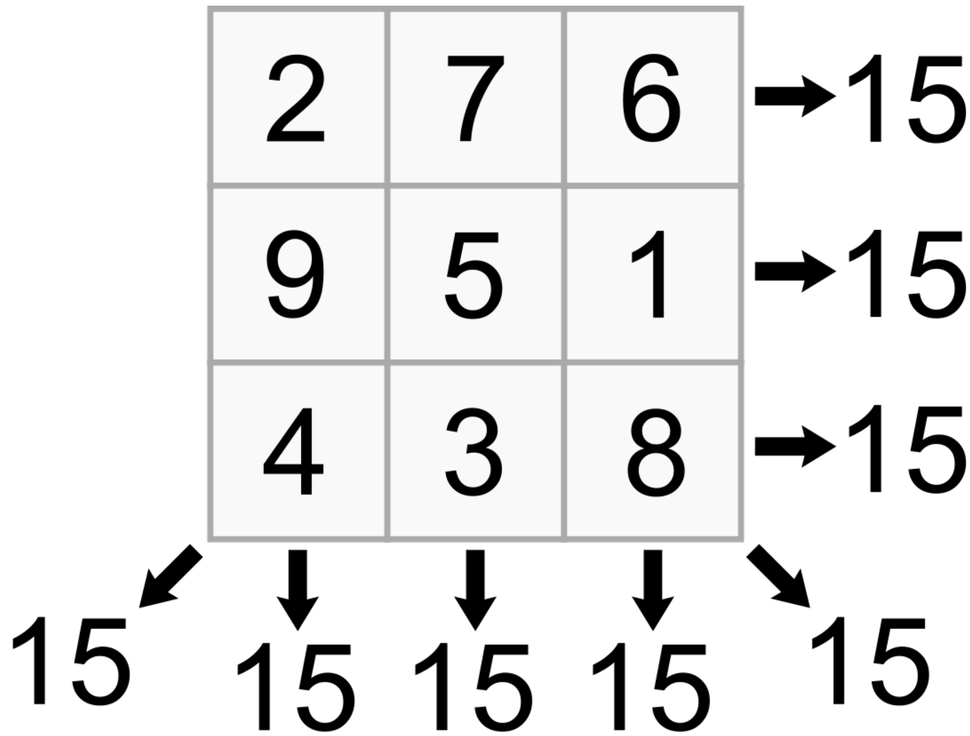
Problem 1. Play the 1-9 game with the students sitting next to you. Use the following grid to play (since we don't actually have a deck of cards). On your turn, write your initials in one of the boxes that are empty, and then check if you won the game. *Make sure to use pencil so that you can erase your writing and play multiple times.*

1	2	3	4	5	6	7	8	9

Problem 2. What is the result of the 1-9 game if both players play optimally? Should player 1 always win? Should player 2 always win? Or will it always be a draw? Is there even a guaranteed result?

2. Magic Squares

Let's take a pause on this game. We are going to learn about a fun construction called a "magic square." Don't worry, you'll see the connection to the game soon enough. A **magic square** of **size** n is the numbers $1, 2, \ldots, n^2$ arranged in a square so that the sum of any row, column, or diagonal is the same (and this sum is called the **magic number**). Here's an example:



Problem 3. What is the size of the above magic square? What is the magic number?

We will learn how to build our own magic squares later. For now, we will investigate properties of magic squares.

Problem 4. For this problem, we are *only* considering magic squares of size 3. Can you create a magic square with magic number that is not 15? Why or why not?

3. Building a Magic Square

Let's figure out how to build a magic square.

Problem 5. To start, let's write all the ways to make 15. But we have a couple important conditions: we can only use the numbers 1 through 9, we can't repeat numbers, and we have to use exactly 3 numbers. Write all the ways you can add up to 15 under those conditions. So for example, we want to count $1+5+9=15$ but not $1+1+13=15$. (We call these ways of breaking up 15 **partitions**.)

Hint: there should be 8 *ways. Keep in mind that* $1+5+9$ *and* $9+5+1$ *are considered the same.*

Problem 6. Look at the list of ways to make 15 you found in Problem 5. For each number 1 through 9, count how many times they appear in the list. Fill out the following chart:

(1)

(2)

(3)

(4)

(5)

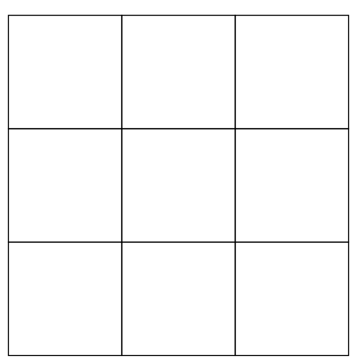
(6)

(7)

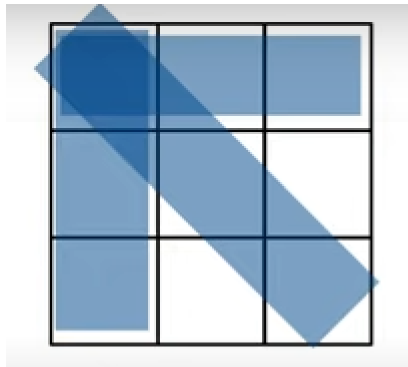
(8)

(9)

Problem 7. For a magic square to be a magic square, we need every row, column, and diagonal to sum to the same value. For each square in the following figure, write how many sums it is involved in (write it in the cell).



For example, the top-left square is involved in 3 sums, illustrated below.



Problem 8. Based on what we just found in the previous problems, what number must go in the centre of the magic square? Once you have figured it out, fill it in below.

Problem 9. Continuing on, what numbers can go in the top-left corner of your magic square? Pick one and write it in the top-left corner of the magic square above. Fill in the bottom-right corner as well (what number must go there?).

Problem 10. Fill in the other two corners of the magic square.

Problem 11. Finish filling in your magic square. How many ways are there to construct a magic square of size 3?

4. Why Magic Squares?

You might be wondering, what is the point of a magic square? Why are we learning about them? You will see soon enough.

Problem 12. Play the 1-9 game on the following grid. The game should work exactly as it did before. Do you notice anything when a player wins? Which 3 cards did they use to win?

2	7	6
9	5	1
4	3	8

Problem 13. Recall that earlier we wrote down all the ways to make 15 from 3 distinct cards. Draw each of these partitions on the magic square above (eg. for $1 + 5 + 9 = 15$, draw a straight line through $9 - 5 - 1$). Do you notice anything about these partitions?

Problem 14. We call two games **isomorphic** if they are played the same, even if they look different. For example, you can play Nim with toothpicks or with a stack of coins, but the gameplay is still the same, so we would call toothpick Nim **isomorphic** to coin Nim. Try to think of another game that you can play on a 3-by-3 grid that works similar to the game we have been playing. How do you win that game? Are these games isomorphic?

Problem 15. Is the 1-9 game a win for Player 1, Player 2, or a draw? Why?

5. Challenge Problems

Problem 16. Let's play the 1-16 game. It's the exact same as the 1-9 game, except the cards are numbered 1-16 and the winner must have 4 cards that sum to 34. Is the 1-16 game isomorphic to a larger version of the game you found in Problem 14?

Problem 17. Construct a magic square of size 3 but using a different set of numbers. Pick your own set of numbers and see if you can build your own magic square. Keep trying until you successfully make a magic square. If you play the 1-9 game but with the numbers in your magic square (and where the goal is your magic number for your magic square), who will win? Or will it be a draw?