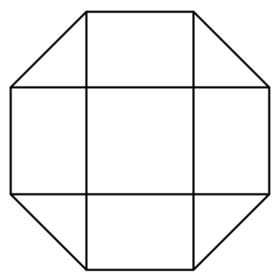
ORMC AMC Group: Week 6
Geometry: Triangles

October 30, 2022

1 Geometric Probability (Review)

1.1 Examples Solutions

1. **(2011 AMC 10B #16)** A dart board is a regular octagon divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?



WLOG, the side length of the octagon is 1.

Then, the total area of the octagon is $1 + 4 \cdot \frac{1}{\sqrt{2}} + 4\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2$, and the area of the center square is 1. So, the probability that we land in the center is

$$\frac{1}{1 + 4 \cdot \frac{1}{\sqrt{2}} + 4\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{1+2\sqrt{2}+1} = \frac{1}{2+2\sqrt{2}} = \frac{2\sqrt{2}-2}{4} = \boxed{\frac{\sqrt{2}-1}{2}.}$$

1.2 Exercises Solutions

1. **(2011 AMC 10B #13)** Two real numbers are selected independently at random from the interval $[-20, 10]$. What is the probability that the product of those numbers is greater than zero?

   Note that the first number is positive with probability $1/3$, and is negative with probability $2/3$. We need both numbers to have the same sign for their product to be positive, so this happens with probability $\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{9} + \frac{4}{9} = \boxed{\frac{4}{9}}$

2. **(1998 AIME #9)** Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

We may represent the possible pairs of times when they arrive as the unit square in the first quadrant. Then, the times when they will see each other are all points satisfying $x-m \leq y \leq x+m$. This means that the probability that they *do not* see each other is $(1-m)^2$, the area within the unit square which does not satisfy the above inequalities. Thus, we have $(1-m)^2 = 0.6 = \frac{3}{5}$. So m in hours is $1-\sqrt{\frac{3}{5}}$, which means that m in minutes is $60-60\sqrt{\frac{3}{5}} = 60-12\sqrt{5}\sqrt{3} = 60-12\sqrt{15}$. So, the answer is $60+12+15 = \boxed{87}$.

3. **(2004 AIME I #10)** A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Given that the probability that the circle will not touch diagonal AC is m/n, where m and n are relatively prime positive integers. Find $m+n$.

   By far the easiest way to do this is coordinate geometry. In general, unless you immediately see a quick "pure"-geometric way to solve a geometric probability problem, coordinate geometry gives an easy way to solve it. This applies to geometry in general– if you don't know how to solve a problem using just geometry theorems, turning it into coordinate geometry makes it an algebra problem instead of a geometry problem.

   We can consider the original area of the rectangle to be defined by the inequalities $0 \leq x \leq 36$ and $0 \leq y \leq 15$. Then, we split it in two with the diagonal, so $0 \leq y \leq \frac{15}{36}x$. Then, within this triangle, the center of the circle must be one unit away from each side. So, we have that the center of the circle may lie within the area defined by $x \leq 35$ and $1 \leq y \leq \frac{15}{36}x - \frac{39}{36}$. The vertices of this area are $(35,1), (35, \frac{35\cdot 15}{36} - \frac{39}{36}), (\frac{36}{15}(1+\frac{39}{36}), 1)$
   Simplifying these points gives us:

$$(35,1)$$

$$(35, \frac{27}{2})$$

$$(5,1)$$

   And applying the shoelace theorem gives us that the area is

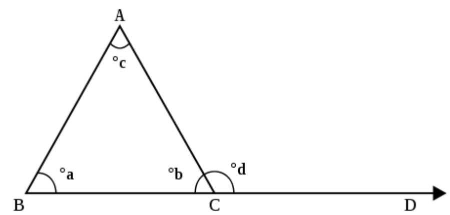
$$\frac{1}{2}\left|\left((\frac{35\cdot 27}{2} + 35 + 5) - (35 + \frac{27\cdot 5}{2} + 35)\right)\right| = \frac{1}{2}(375)$$

   Since we have a symmetrical area above the diagonal, the total area we are allowed to be in is 375. And, the total area we are able to be in is 1 unit away from the top, bottom, and sides, giving us a full 13×34 rectangle of 442 units. This means the probability the circle does not cross the diagonal is $\frac{375}{442}$, which is in simplest form so our answer is $375 + 442 = \boxed{817}$.

2 Triangle Basics

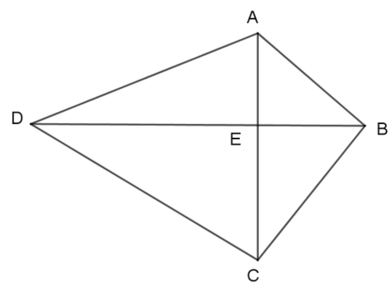
2.1 Exercises Solutions

1. **F**ind d in terms of only a, b, and c:



Note that $b + d = 180 = a + b + c$, so $d = a + c$.

2. **W**hat is the area of quadrilateral $ABCD$ if $AC = 10$, $BD = 25$, and $\angle AED$ is a right angle?



We have two triangles ADB and CDB. The base of both of them is 25, and the height of ADB is AE, while the height of CDB is $10-AE$. The total area of both is then $\frac{1}{2}(25\cdot AE+25(10-AE)) = \frac{1}{2}(25 \cdot 10) = \boxed{125}$.

3. **W**hat is the area of a triangle with side lengths $13, 14, 15$?

We will use Heron's formula. The semiperimeter of the triangle is 21, so the area is

$$\sqrt{21 \cdot 6 \cdot 7 \cdot 8} = \sqrt{3 \cdot 7 \cdot 3 \cdot 2 \cdot 7 \cdot 2^3} = 3 \cdot 7 \cdot 2^2 = \boxed{84}.$$

Note also that such a triangle can be formed by adjoining $9-12-15$ and $5-12-13$ right triangles along the length 12 side. So, it is a triangle with base 14 and height 12, giving it area $7 \cdot 12 = 84$.

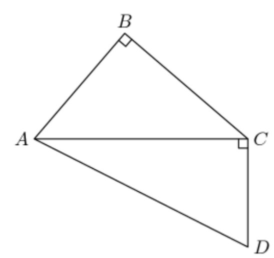
3 Pythagorean Theorem

3.1 Examples Solutions

1. Two sides of a right triangle have the lengths 4 and 5. What is the product of the possible lengths of the third side?

   The third side may be a leg, in which case the hypotenuse is 5 and it has length $\sqrt{5^2 - 4^2} = 3$, or it may be the hypotenuse, in which case it has length $\sqrt{5^2 + 4^2} = \sqrt{41}$. The product of these values is $3\sqrt{41}$.

2. In quadrilateral ABCD, angle B is a right angle, diagonal AC is perpendicular to CD, AB=18, BC=21, and CD=14. Find the perimeter of ABCD.

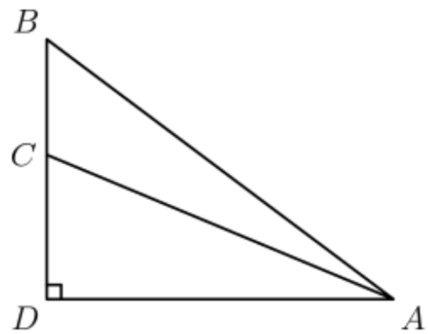
   

   We apply the pythagorean theorem twice. First, we find that $AC^2 = 18^2 + 21^2 = 324 + 441 = 765$. Then, we have that $AD^2 = 765 + 14^2 = 765 + 169 = 961 = 900 + 60 + 1 = (30 + 1)^2$ (also note that 961 has digits in the reverse order of 169, which is the square of 13, but this doesn't work in general).

   So, the perimeter is $18 + 21 + 14 + 31 = 31 + 21 + 32 = \boxed{84}$.

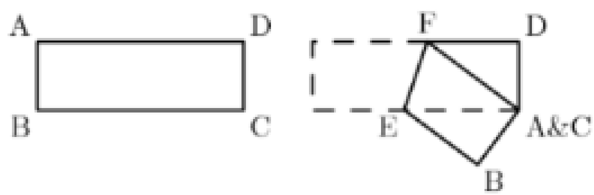
3.2 Exercises Solutions

1. What is the area of ABC if $AC = 13, AB = 15$, and $DC = 5$?



Note that ACD is a $5-12-13$ triangle, so $AD = 12$, which means that ABD is a $3-4-5$ $(9-12-15)$ triangle so $DB = 9$. Thus, we have ABC has base $9-5=4$ and height 12, so its area is $\boxed{24}$.

2. In rectangle ABCD, AB = 3 and BC = 9. The rectangle is folded so that points A and C coincide, forming the pentagon ABEFD. What is the length of segment EF? Express your answer in simplest radical form.



For convenience, we will relabel $A\&C$ by G. Then, by the pythagorean theorem, we have $FG^2 = FD^2 + DG^2$, where $FG + FD = 9$. So letting $x = FD$, we may rewrite this as $(9-x)^2 = x^2 + 3^2 \implies 81 - 6x = 9 \implies x = 4$ The same applies for triangle EGB, so we have $EB = 4$ as well.

We may draw a perpendicular from E to FG, which intersects FG at P. The length of EP is 3 since we have that $EPGB$ is a rectangle, and since $EB = 4$ and $FG = 5$, it follows that $FP = 1$.

Thus, we have that the length of EF is $\sqrt{3^2 + 5 - 4^2} = \sqrt{9+1} = \boxed{\sqrt{10}}$.

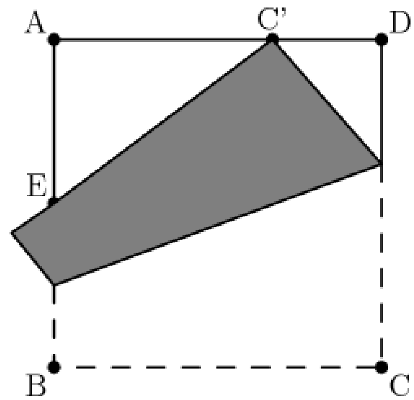
3. **(2021 AMC 10A #13)** What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?

Notice that by applying the pythagorean theorem, all the triangles ABC, ABD, and ACD are right triangles, with right angles at A. In particular, this means that if we set ABC as the base, then the height of the tetrahedron is $AD = 4$.

The volume of a tetrahedron is $\frac{1}{3}Bh$, where B is the area of the base, and h is the height. For our tetrahedron, we have $B = \frac{1}{2} \cdot 2 \cdot 3$ and $h = 4$, so the total volume is $\boxed{4}$.

4. **(2021 AMC 10B #21)** A square piece of paper has side length 1 and vertices A, B, C, and D in that order. As shown in the figure, the paper is folded so that vertex C meets edge $\overline{AD}$ at point C', and edge $\overline{BC}$ intersects edge $\overline{AB}$ at point E. Suppose that $C'D = \frac{1}{3}$. What is the perimeter of triangle $\triangle AEC'$?

We can set the point on CD where the fold occurs as point F. Then, we can set FD as x, and CF as $1-x$ because of symmetry due to the fold. It can be recognized that this is a right triangle, and solving for x, we get,



$$x^2 + \left(\frac{1}{3}\right)^2 = (1-x)^2 \to x^2 + \frac{1}{9} = x^2 - 2x + 1 \to x = \frac{4}{9}$$

We know this is a 3-4-5 triangle because the side lengths are $\frac{3}{9}, \frac{4}{9}, \frac{5}{9}$. We also know that EAC' is similar to $C'DF$ because angle $EC'F$ is a right angle. Now, we can use similarity to find out that the perimeter is just the perimeter of $C'DF \times \frac{AC'}{DF}$. That's just $\frac{4}{3} \times \frac{\frac{2}{3}}{\frac{4}{9}} = \frac{4}{3} \times \frac{3}{2} = 2$. Therefore, the final answer is $\boxed{2}$

4 Trigonometric Functions

4.1 Examples Solutions

1. In $\triangle ABC$, we have $AB = 13$, $BC = 14$, and $AC = 15$. Point P lies on BC, and $AP \perp BC$. What is the length of BP?

   This is a perfect opportunity to use law of cosines. By the definition of cosine, we have that $BP = BA\cos(B) = 13\cos(B)$. By the law of cosines, we have:

   $$\cos(B) = \frac{13^2 + 14^2 - 15^2}{2(13)(14)} = \frac{5}{13}$$

   which means that $BP = \boxed{5}$.

2. In $\triangle ABC$, we have $AB = 13$, $\angle A = 75°$, and $\angle B = 45°$. What are the perimeter and area of $\triangle ABC$? (Hint: $\sin(75°) = \frac{\sqrt{2}+\sqrt{6}}{4}$)

   We can use law of sines. We have $C = 60°$, so

   $$\frac{\sin 45}{AC} = \frac{\sin 60}{13} \implies AC = \frac{13\sin 45}{\sin 60} = \frac{13/\sqrt{2}}{\sqrt{3}/2} = \frac{13\sqrt{2}}{\sqrt{3}} = \frac{13\sqrt{6}}{3}$$

   Using our area formula, we get that the area is:

   $$\frac{1}{2}13 \cdot \frac{13\sqrt{6}}{3} \cdot \sin(75°) = \frac{1}{2} \cdot \frac{13^2}{3} \cdot \frac{2\sqrt{3}+6}{4}.$$

   In order to find the perimeter, we just need to find BC, which we can again do with law of sines. We have

   $$\frac{\sin 75}{BC} = \frac{\sin 60}{13} \implies AC = \frac{13\sin 75}{\sin 60} = \frac{13 \cdot 4/(\sqrt{2}+\sqrt{6})}{\sqrt{3}/2} = \frac{13 \cdot 8}{\sqrt{6}+3\sqrt{2}} = \frac{13 \cdot 8 \cdot (3\sqrt{2}-\sqrt{6})}{12}$$

   So, the perimeter of the triangle is:

   $$13 + \frac{13\sqrt{6}}{3} + \frac{13 \cdot 8 \cdot (3\sqrt{2}-\sqrt{6})}{12}.$$

4.2 Exercises Solutions

1. In $\triangle ABC$, $\angle B = 3\angle C$. If $AB = 10$ and $AC = 15$, compute the length of BC.

   I will preface this by saying that in order to solve this, we need some knowledge of trigonometric identities, which we have not gone over yet. Besides that, it is a reasonable application of law of sines and law of cosines.

   By law of sines, we have:

   $$\frac{15}{\sin(3C)} = \frac{10}{\sin(C)} \implies \frac{3}{2} = \frac{\sin(3C)}{\frac{(}{C})}$$

   By the angle addition formulas for sine and cosine, we have:

   $$\begin{aligned} \sin(3C) &= \sin(2C)\cos(2C) + \sin(C)\cos(2C) \\ &= 2\sin(C)\cos^2(C) + 2\sin(C)\cos^2(C) - \sin(C) \\ &= 4\cos^2(C)\sin(C) + \sin(C) = \sin(C)(4\cos^2(C) - 1) \\ &\implies \frac{3}{2} = 4\cos^2(C) - 1. \end{aligned}$$

Then, note that if we know $\cos(A)$, we can find BC by using law of cosines. By the symmetry of cosine about $180°$, we have $\cos(A) = \cos(180 - 3C - C) = \cos(180 - 4C) = \cos(4C)$. Then, we apply the cosine angle addition (double angle) identities:

$$\cos(4C) = 2\cos^2(2C) - 1 = 2(2\cos^2(C) - 1) - 1 = 4\cos^2(C) - 2 - 1 = \frac{3}{2} - 1 = -\frac{1}{2}$$

Finally, applying law of cosines, we get:

$$\begin{aligned} BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos(A) &= 10^2 + 15^2 - 2 \cdot 10 \cdot 15\cos(4C) \\ &= 100 + 225 - 60\frac{-1}{2} = 325 + 30 = 355. \end{aligned}$$

So, the length of BC is $\sqrt{355}$.

2. **(2019 AMC 12A #19)** In $\triangle ABC$ with integer side lengths, $\cos A = \frac{11}{16}$, $\cos B = \frac{7}{8}$, and $\cos C = -\frac{1}{4}$. What is the least possible perimeter for $\triangle ABC$?

   Notice that by the Law of Sines, $a : b : c = \sin A : \sin B : \sin C$, so let's flip all the cosines using $\sin^2 x + \cos^2 x = 1$ (mentioned in class).

   $\sin A = \frac{3\sqrt{15}}{16}$, $\quad \sin B = \frac{\sqrt{15}}{8}$, $\quad$ and $\quad \sin C = \frac{\sqrt{15}}{4}$

   These are in the ratio $3 : 2 : 4$, so our minimal triangle has side lengths 2, 3, and 4 $\implies \boxed{9}$.

3. **(2001 AIME #4)**. In triangle ABC, angles A and B measure $60°$ and $45°$, respectively. The bisector of angle A intersects BC at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

   After chasing angles, $\angle ATC = 75°$ and $\angle TCA = 75°$, meaning $\triangle TAC$ is an isosceles triangle and $AC = 24$.

   Using law of sines on $\triangle ABC$, we can create the following equation:

   $\frac{24}{\sin(\angle ABC)} = \frac{BC}{\sin(\angle BAC)}$

   $\angle ABC = 45°$ and $\angle BAC = 60°$, so $BC = 12\sqrt{6}$.

   We can then use the Law of Sines area formula $\frac{1}{2} \cdot BC \cdot AC \cdot \sin(\angle BCA)$ to find the area of the triangle.

   $\sin(75)$ can be found through the sin addition formula.

   $\sin(75) = \frac{\sqrt{6}+\sqrt{2}}{4}$

   Therefore, the area of the triangle is $\frac{\sqrt{6}+\sqrt{2}}{4} \cdot 24 \cdot 12\sqrt{6} \cdot \frac{1}{2}$

   $72\sqrt{3} + 216$

   $72 + 3 + 216 = \boxed{291}$

4. **(2017 AMC 10B #19)** Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

   Note that by symmetry, $\triangle A'B'C'$ is also equilateral. Therefore, we only need to find one of the sides of $A'B'C'$ to determine the area ratio. WLOG, let $AB = BC = CA = 1$. Therefore, $BB' = 3$ and $BC' = 4$. Also, $\angle B'BC' = 120°$, so by the Law of Cosines, $B'C' = \sqrt{37}$. Therefore, the answer is $(\sqrt{37})^2 : 1^2 = \boxed{37 : 1}$