**OLGA RADKO MATH CIRCLE: ADVANCED 3**

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# Worksheet 4: Groups and the Rubik's cube

In this worksheet, we will consider two kind of Rubik's cubes. The classic $3 \times 3 \times 3$ Rubik's cube and the smaller version $2 \times 2 \times 2$ Rubik's cube. The latter is also known as the "Pocket Cube".

**Problem 4.0:** In what follows, we will write $\mathcal{G}_{\text{pocket}}$ for the group of the $2 \times 2 \times 2$ Rubik's cube.
Show that the group $\mathcal{G}_{\text{pocket}}$ is a subgroup of $\mathcal{G}_{\text{rubik}}$.
Show that the group of the $2 \times 2 \times 2$ Rubik's cube can not be generated by a single element.
Can the group $\mathcal{G}_{\text{pocket}}$ be generated by 2 elements?
**Solution 4.0:**

Permutations are going to be useful to solve the pocket cube. First, we will work in the following problem related to permutations.

**Problem 4.1:** The group $S_n$ is the group of permutations of a set with $n$ elements. For instance, the group $S_2$ is the set of permutations of $\{1, 2\}$. There are only two such permutations: the identity and $(12)$. Thus, we conclude that

$$S_2 \simeq \langle x \mid x^2 \rangle.$$

Find the number of elements of the group $S_n$.
Find generators and relations for the group $S_n$.

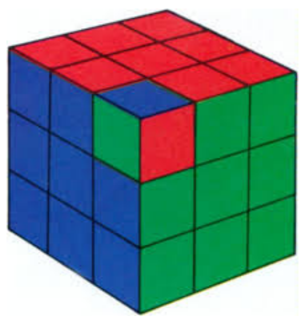
**Solution 4.1:**

**Problem 4.2:** A "cubie" is a corner cube of the $2 \times 2 \times 2$ Rubik's cube (or Pocket Cube).
Show that given a Pocket Cube and two of its cubies; say $C_1$ and $C_2$, we can find a sequence of movements that swaps $C_1$ with $C_2$ and leave all other cubies intact.
Conclude that $S_8$ is a subgroup of $\mathcal{G}_{\text{pocket}}$.
**Solution 4.2:**

**Problem 4.3:** Show that the group of rotations of a cube has 24 elements.
Find the generators and relations between its elements.
**Solution 4.3:**

**Definition 1.** A *twist* of a cubie is a permutation that swaps the three colors of the cubie.
A twist is just an abstract permutation that may not be possible to perform in reality. For instance, if we were able to twist the colors of a single cubie, then the cube could not be solved.

The picture below shows an example of a $3 \times 3 \times 3$ Rubik's cube with a single twisted cubie.



Although the Rubik's cube in the picture can not be solved, the concept of twist may help us to count the number of possible permutations of the cube and to understand the group structure of both $\mathcal{G}_{\text{rubik}}$ and $\mathcal{G}_{\text{pocket}}$.

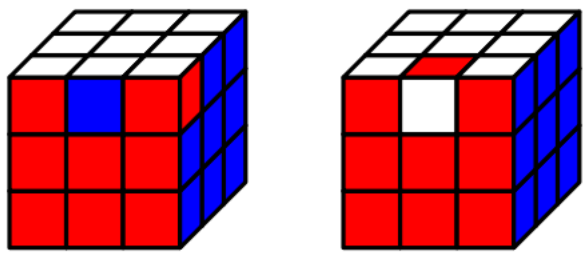
**Problem 4.4:** Show that if we twist 7 of the 8 cubies of the Pocket Cube, there is a unique twist of the 8th cubie for which the obtained Pocket Cube can be solved.
**Solution 4.4:**

**Problem 4.5:** Find the number of possible permutations of the Pocket Cube.
**Solution 4.5:**

In the case of $3 \times 3 \times 3$ Rubik's cube, we have many more "cubies". The central cube of each face can be made to be fixed when we solve the cube. We will call cubies only the "little cubes" that are actually moving when we try to solve the cube.

The $3 \times 3 \times 3$ Rubik's cube has 8 *corner cubies* and 12 *edge cubies*.

A twist of the corner cubie is the same as in the Pocket Cube. A twist of an edge cubie is simply exchanging its two colors as shown in the picture below.



Again, if we only apply a twist to a single edge cubie, then the obtained cube cannot be solved. A sequence of twists is said to be a *legal twist* if the obtained cube can be solved.

We will write $\mathcal{G}_O \leqslant \mathcal{G}_{\text{rubik}}$ for the subgroup obtained by legal twists. We write $\mathcal{G}_P \leqslant \mathcal{G}_{\text{rubik}}$ for the group of transformations of the cube that permutes the cubies but preserves their orientation.

**Problem 4.6:** Write three examples of elements of $\mathcal{G}_O$.
Write three examples of elements of $\mathcal{G}_P$.
Show that $\mathcal{G}_O \cap \mathcal{G}_P = \{1\}$, i.e., the only common element of these subgroups is the identity.
Show that for every element $g \in \mathrm{G}_{\text{rubik}}$, we have that

$$g^{-1}\mathcal{G}_O g = \{g^{-1}hg \mid h \in \mathcal{G}_O\} = \mathcal{G}_O.$$

**Solution 4.6:**

**Definition 2.** Let $G$ be a group and $N$ be a subgroup. We say that $N$ is a *normal subgroup* of $G$ if for every $g \in G$, we have that $g^{-1}Ng \subset N$. In this case, we write $N \trianglelefteq G$.

**Definition 3.** Let $(G_1, \otimes_1)$ and $(G_2, \otimes_2)$ be two groups. The *direct product* $G_1 \times G_2$ is the set of pairs

$$\{(g_1, g_2) \mid g_1 \in G_1 \text{ and } g_2 \in G_2\}.$$

This set is naturally endowed with a binary operation:

$$(g_1, g_2) \times (h_1, h_2) = (g_1 \otimes_1 h_1, g_2 \otimes_2 h_2).$$

**Problem 4.7:** Let $(G_1, \otimes_1)$ and $(G_2, \otimes_2)$. Show that $(G_1 \times G_2, \times)$ is a group.
What is the inverse of an element in this group?
What is the identity element?
Consider $(\mathbb{R} \times \mathbb{R}, \times)$ where we consider $\mathbb{R}$ with the multiplicative operation. Explain the group law of this product.
**Solution 4.7:**

**Definition 4.** Let $(G, *)$ and $(H, \otimes)$ be two groups. A *group homomorphism* is a function $\psi \colon G \to H$ that satisfies the following rules:

- $\psi(1_G) = 1_H$, and
- for every $g_1, g_2 \in G$, we have that $\psi(g_1 * g_2) = \psi(g_1) \otimes \psi(g_2)$.

In other words, a group homomorphism a function that "respects" the structure of both groups.
We say that a group homomorphism is a *group isomorphism* if $\psi$ is one-to-one and onto. If there exists a group isomorphism between $G$ and $H$, we say that these groups are isomorphic, and we write $G \simeq H$ (the symbol $\simeq$ is read as "isomorphic to").

**Problem 4.8:** Show that $\mathcal{G}_O \simeq (\mathbb{Z}_3^7) \times (\mathbb{Z}_2)^{11}$.
In order to do so, you can show that given twists of 11 of the edge cubies and twists of 7 of the corner cubies, there is a unique twist of the last edge cubie and a unique twist of the last corner cubie that will make the twist legal.
**Solution 4.8:**

**Definition 5.** A *transposition* in $S_n$ is a permutation that only swaps two elements, i.e., it has the form $(i, j)$.

**Problem 4.9:** Show that every element of $S_3$ can be written as a product of transpositions.
Show that every element of $S_4$ can be written as a product of transpositions.
Show that every element of $S_n$ can be written as a product of transpositions.
**Solution 4.9:**

**Definition 6.** The *alternating group*, usually denoted by $A_n$, is the subgroup of $S_n$ generated by all the permutations that can be written as an even product of transpositions.

**Problem 4.10:** Find the order (number of elements) of the group $A_8$.
**Solution 4.10:**

**Definition 7.** Let $G$ be a group. We say that $G$ is the *semi-direct* product of $N$ and $K$, written $G = N \rtimes K$ if the following conditions are satisfied:

- $N \trianglelefteq G$,
- $G = NK = \{g \mid g = nk \text{ with } n \in N, k \in K\}$, and
- $N \cap K = \{1_G\}$.

**Problem 4.11:** Show that

$$\mathcal{G}_P \simeq (A_8 \times A_{12}) \rtimes \mathbb{Z}_2.$$

**Hint:** The group $A_8$ corresponds to permutations of corner cubies, the group $A_{12}$ corresponds to permutation of edge cubies, and $\mathbb{Z}_2$ corresponds to a single permutation that exhanges two edge cubies and two corner cubies.
**Solution 4.11:**

**Problem 4.12:** Prove that

$$\mathcal{G}_{\text{rubik}} := (\mathbb{Z}_3^{11} \times \mathbb{Z}_2^{11}) \rtimes ((A_8 \times A_{12}) \rtimes \mathbb{Z}_2).$$

**Solution 4.12:**

**Problem 4.13:** Find the exact number of possible permutations of the $3 \times 3 \times 3$ Rubik's cube.
**Solution 4.13:**

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