ORMC AMC Group: Week 6
Geometry: Triangles

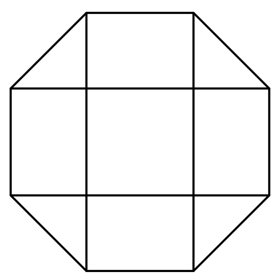
October 30, 2022

1 Geometric Probability (Review)

Geometric probability is a method of calculating probability by using length, area, or volume. We may use this technique not only for problems explicitly involving geometry, but also for probability questions with *continuous* variables. It is usually possible to simplify a continuous probability problem by viewing it as a description of a geometric figure, and using geometric probability techniques.

1.1 Examples

1. **(2011 AMC 10B #16)** A dart board is a regular octagon divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?



1.2 Exercises

1. **(2011 AMC 10B #13)** Two real numbers are selected independently at random from the interval $[-20, 10]$. What is the probability that the product of those numbers is greater than zero?

2. **(1998 AIME #9)** Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

3. **(2004 AIME I #10)** A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Given that the probability that the circle will not touch diagonal AC is m/n, where m and n are relatively prime positive integers. Find $m + n$.

2 Triangle Basics

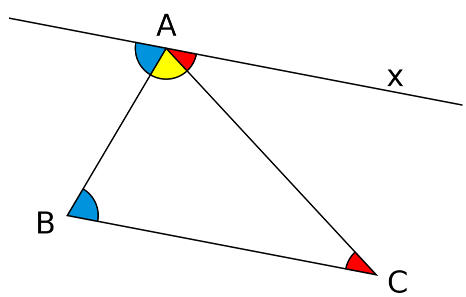
Perimeter: The perimeter P of a triangle is the sum of the lengths of its sides. Another important related value is the *semiperimeter*, often denoted s, which is simply $P/2$.

Area: The most well-known area formula for triangle area is $\frac{1}{2}bh$, where b is the length of the base of a triangle, and h is the height corresponding to that base. This formula can be derived from the corresponding formula for a parallelogram, since any triangle can be doubled to create a parallelogram. Another useful formula is called *Heron's Formula*:

$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

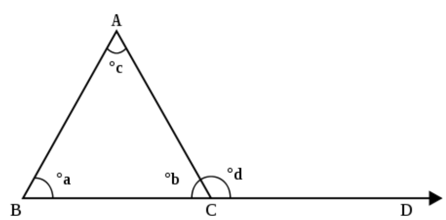
where a, b, c are the lengths of the sides of the triangle, and s is the semiperimeter.

Angles: The sum of the internal angles of a triangle is always 180°. This can be demonstrated by using parallel line/transversal theorems:

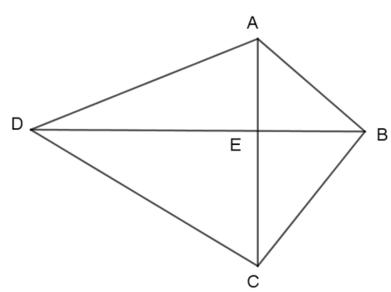


2.1 Exercises

1. **F**ind d in terms of only a, b, and c:



2. **W**hat is the area of quadrilateral $ABCD$ if $AC = 10$, $BD = 25$, and $\angle AED$ is a right angle?

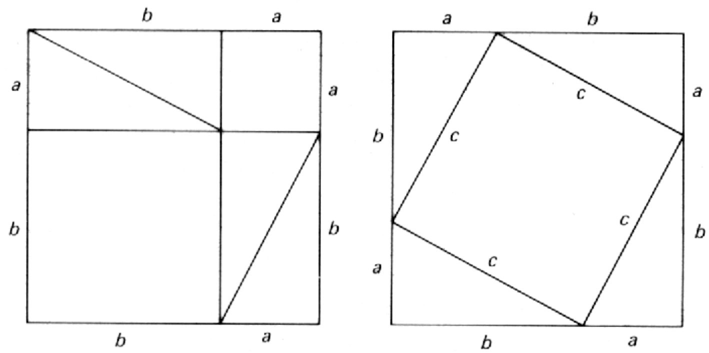


3. **W**hat is the area of a triangle with side lengths $13, 14, 15$?

3 Pythagorean Theorem

A triangle is a right triangle if one of its internal angles has measure $90°$. Note that only one angle may have measure $\geq 90°$, since two such angles would add to more than $180°$.

The most famous theorem related to right triangles is the ***Pythagorean Theorem***, which states that a triangle is a right triangle if and only if its sides a, b, c, WLOG $a \leq b \leq c$, satisfy $a^2 + b^2 = c^2$. This is usually proved using the following type of diagram:



Note that we can extend this result. If c is the length of the longest side of a triangle, then its corresponding angle C is the largest angle, and:

- $c^2 < a^2 + b^2$ precisely when the triangle is acute. This is because c is smaller than the hypotenuse of a right triangle with legs a and b, if and only if C is less than $90°$.
- $c^2 > a^2 + b^2$ precisely when the triangle is obtuse. We know c is longer than the hypotenuse of a right triangle with legs a and b if and only if C is greater than $90°$.

Integer triples a, b, c that satisfy $a^2 + b^2 = c^2$ are called pythagorean triples, and they show up a lot. Here are a few that are good to know:

$$(3, 4, 5) \quad (5, 12, 13) \quad (7, 24, 25) \quad (8, 15, 17) \quad (9, 40, 41) \quad (11, 60, 61) \quad (12, 35, 37)$$

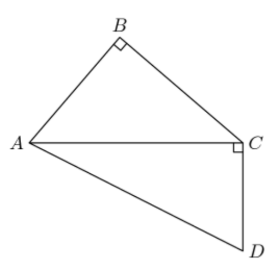
Additionally, there are a couple special right triangles, which are not pythagorean triples, but are important to know:

- Isosceles Right Triangle (45-45-90): the sides are in the ratio $1 : 1 : \sqrt{2}$, since we have $a^2 + a^2 = c^2$
- 30-60-90 Right Triangle: the sides are in the ratio $1 : \sqrt{3} : 2$. (note that $1 < \sqrt{3} < 2$.)

3.1 Examples

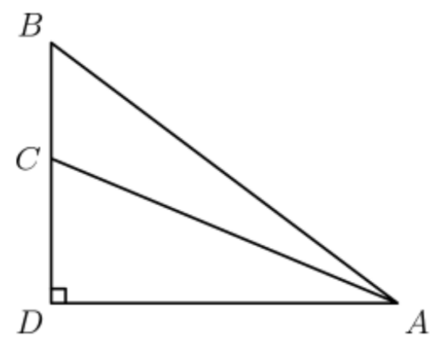
1. Two sides of a right triangle have the lengths 4 and 5. What is the product of the possible lengths of the third side?

2. In quadrilateral ABCD, angle B is a right angle, diagonal AC is perpendicular to CD, AB=18, BC=21, and CD=14. Find the perimeter of ABCD.

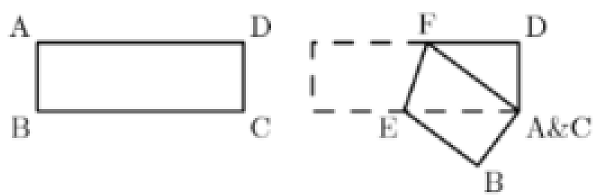


3.2 Exercises

1. What is the area of ABC if $AC = 13, AB = 15$, and $DC = 5$?

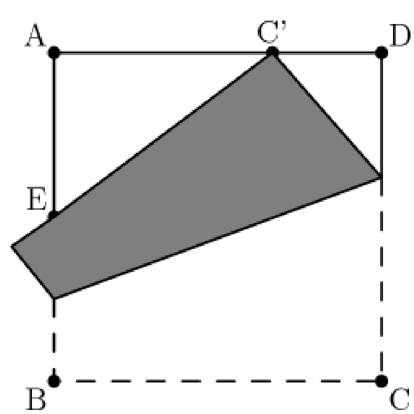


2. In rectangle ABCD, AB = 3 and BC = 9. The rectangle is folded so that points A and C coincide, forming the pentagon ABEFD. What is the length of segment EF? Express your answer in simplest radical form.



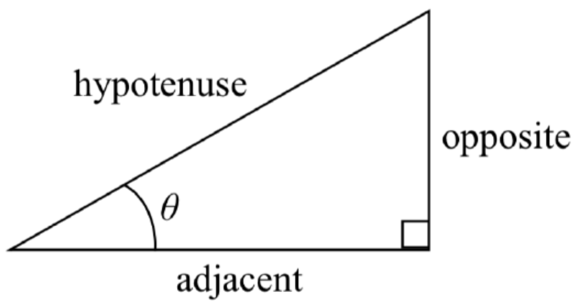
3. **(2021 AMC 10A #13)** What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?

4. **(2021 AMC 10B #21)** A square piece of paper has side length 1 and vertices A, B, C, and D in that order. As shown in the figure, the paper is folded so that vertex C meets edge $\overline{AD}$ at point C', and edge $\overline{BC}$ intersects edge $\overline{AB}$ at point E. Suppose that $C'D = \frac{1}{3}$. What is the perimeter of triangle $\triangle AEC'$?



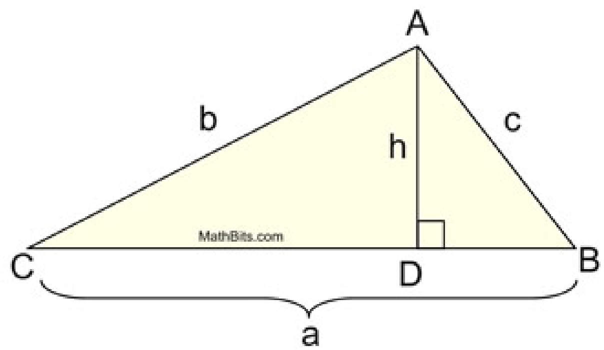
4 Trigonometric Functions

The main trigonometric functions we will work with are called *sine*(sin), *cosine*(cos), and *tangent*(tan). They are functions of angles, and are defined on a right triangle as follows:



$$\sin(\theta) = \frac{opposite}{hypotenuse}, \quad \cos(\theta) = \frac{adjacent}{hypotenuse}, \quad \tan(\theta) = \frac{opposite}{adjacent}$$

Note that this means $opposite = (hypotenuse) \cdot \sin(\theta)$, which gives us a new way to calculate area:



$$h = b \cdot \sin(C) \implies K = \boxed{\frac{1}{2}a \cdot b\sin(C)}$$

So, for any two sides of a triangle (x, y), and the angle between them (Z), the area is $\frac{1}{2}xy\sin(Z)$. We can write this out for each pair of sides in the diagram above:

$$K = \frac{1}{2}ab\sin(C) = \frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B)$$

If we multiply every one of these expressions by $\frac{2}{abc}$, we get the (extended) ***Law of Sines***:

$$\frac{2K}{abc} = \boxed{\frac{\sin(C)}{c} = \frac{\sin(A)}{a} = \frac{\sin(B)}{b}}.$$

Another thing we can try with this triangle diagram is applying the pythagorean theorem, since h creates two right triangles. We have:

$$b^2 = (CD)^2 + h^2, \quad c^2 = (a - CD)^2 + h^2 = a^2 - 2a(CD) + (CD)^2 + h^2$$

Subtracting the first equation from the second to get eliminate $(CD)^2 + h^2$, we get the ***Law of Cosines***:

$$c^2 - b^2 = a^2 - 2a(CD) = a^2 - 2ab\cos(C) \implies \boxed{c^2 = a^2 + b^2 - 2ab\cos(C)}.$$

4.1 Examples

1. In $\triangle ABC$, we have $AB = 13$, $BC = 14$, and $AC = 15$. Point P lies on BC, and $AP \perp BC$. What is the length of BP?

2. In $\triangle ABC$, we have $AB = 13$, $\angle A = 75^\circ$, and $\angle B = 45^\circ$. What are the perimeter and area of $\triangle ABC$? (Hint: $\sin(75^\circ) = \frac{\sqrt{2}+\sqrt{6}}{4}$)

4.2 Exercises

1. In $\triangle ABC$, $\angle B = 3\angle C$. If $AB = 10$ and $AC = 15$, compute the length of BC.

2. **(2019 AMC 12A #19)** In $\triangle ABC$ with integer side lengths, $\cos A = \frac{11}{16}$, $\cos B = \frac{7}{8}$, and $\cos C = -\frac{1}{4}$. What is the least possible perimeter for $\triangle ABC$?

3. **(2001 AIME #4)**. In triangle ABC, angles A and B measure 60° and 45°, respectively. The bisector of angle A intersects BC at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

4. **(2017 AMC 10B #19)** Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?