1 Geometric Probability (Review)

Geometric probability is a method of calculating probability by using length, area, or volume. We may use this technique not only for problems explicitly involving geometry, but also for probability questions with continuous variables. It is usually possible to simplify a continuous probability problem by viewing it as a description of a geometric figure, and using geometric probability techniques.

1.1 Examples

1. (2011 AMC 10B #16) A dart board is a regular octagon divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?

![Dart Board Image]

1.2 Exercises

1. (2011 AMC 10B #13) Two real numbers are selected independently at random from the interval $[-20, 10]$. What is the probability that the product of those numbers is greater than zero?

2. (1998 AIME #9) Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly $m$ minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where $a$, $b$, and $c$ are positive integers, and $c$ is not divisible by the square of any prime. Find $a + b + c$.

3. (2004 AIME I #10) A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Given that the probability that the circle will not touch diagonal $AC$ is $m/n$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$. 

1
2 Triangle Basics

**Perimeter:** The perimeter \( P \) of a triangle is the sum of the lengths of its sides. Another important related value is the *semiperimeter*, often denoted \( s \), which is simply \( P/2 \).

**Area:** The most well-known area formula for triangle area is \( \frac{1}{2}bh \), where \( b \) is the length of the base of a triangle, and \( h \) is the height corresponding to that base. This formula can be derived from the corresponding formula for a parallelogram, since any triangle can be doubled to create a parallelogram. Another useful formula is called *Heron’s Formula*:

\[
K = \sqrt{s(s-a)(s-b)(s-c)},
\]

where \( a, b, c \) are the lengths of the sides of the triangle, and \( s \) is the semiperimeter.

**Angles:** The sum of the internal angles of a triangle is always 180°. This can be demonstrated by using parallel line/transversal theorems:

2.1 Exercises

1. Find \( d \) in terms of only \( a, b, \) and \( c \):

2. What is the area of quadrilateral \( ABCD \) if \( AC = 10, BD = 25, \) and \( \angle AED \) is a right angle?

3. What is the area of a triangle with side lengths 13, 14, 15?
3 Pythagorean Theorem

A triangle is a right triangle if one of its internal angles has measure 90°. Note that only one angle may have measure \( \geq 90^\circ \), since two such angles would add to more than 180°.

The most famous theorem related to right triangles is the **Pythagorean Theorem**, which states that a triangle is a right triangle if and only if its sides \( a, b, c \), WLOG \( a \leq b \leq c \), satisfy \( a^2 + b^2 = c^2 \). This is usually proved using the following type of diagram:

Note that we can extend this result. If \( c \) is the length of the longest side of a triangle, then its corresponding angle \( C \) is the largest angle, and:

- \( c^2 < a^2 + b^2 \) precisely when the triangle is acute. This is because \( c \) is smaller than the hypotenuse of a right triangle with legs \( a \) and \( b \), if and only if \( C \) is less than 90°.
- \( c^2 > a^2 + b^2 \) precisely when the triangle is obtuse. We know \( c \) is longer than the hypotenuse of a right triangle with legs \( a \) and \( b \) if and only if \( C \) is greater than 90°.

Integer triples \( a, b, c \) that satisfy \( a^2 + b^2 = c^2 \) are called pythagorean triples, and they show up a lot. Here are a few that are good to know:

\[
(3, 4, 5) \quad (5, 12, 13) \quad (7, 24, 25) \quad (8, 15, 17) \quad (9, 40, 41) \quad (11, 60, 61) \quad (12, 35, 37)
\]

Additionally, there are a couple special right triangles, which are not pythagorean triples, but are important to know:

- Isosceles Right Triangle (45-45-90): the sides are in the ratio 1 : 1 : \( \sqrt{2} \), since we have \( a^2 + a^2 = c^2 \)
- 30-60-90 Right Triangle: the sides are in the ratio 1 : \( \sqrt{3} : 2 \). (note that \( 1 < \sqrt{3} < 2 \).)

3.1 Examples

1. Two sides of a right triangle have the lengths 4 and 5. What is the product of the possible lengths of the third side?

2. In quadrilateral ABCD, angle B is a right angle, diagonal AC is perpendicular to CD, AB=18, BC=21, and CD=14. Find the perimeter of ABCD.
3.2 Exercises

1. What is the area of ABC if \( AC = 13 \), \( AB = 15 \), and \( DC = 5 \)?

2. In rectangle ABCD, \( AB = 3 \) and \( BC = 9 \). The rectangle is folded so that points A and C coincide, forming the pentagon ABEFD. What is the length of segment EF? Express your answer in simplest radical form.

3. (2021 AMC 10A #13) What is the volume of tetrahedron ABCD with edge lengths \( AB = 2 \), \( AC = 3 \), \( AD = 4 \), \( BC = \sqrt{13} \), \( BD = 2\sqrt{5} \), and \( CD = 5 \)?

4. (2021 AMC 10B #21) A square piece of paper has side length 1 and vertices \( A, B, C, \) and \( D \) in that order. As shown in the figure, the paper is folded so that vertex \( C \) meets edge \( AD \) at point \( C' \), and edge \( BC \) intersects edge \( AB \) at point \( E \). Suppose that \( C'D = \frac{1}{3} \). What is the perimeter of triangle \( \triangle AEC' \)?
4 Trigonometric Functions

The main trigonometric functions we will work with are called \( \sin \) (sine), \( \cos \) (cosine), and \( \tan \) (tangent). They are functions of angles, and are defined on a right triangle as follows:

\[
\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}
\]

Note that this means \( \text{opposite} = (\text{hypotenuse}) \cdot \sin(\theta) \), which gives us a new way to calculate area:

\[
h = b \cdot \sin(C) \implies K = \frac{1}{2} a \cdot b \sin(C)
\]

So, for any two sides of a triangle \((x, y)\), and the angle between them \((Z)\), the area is \( \frac{1}{2} xy \sin(Z) \). We can write this out for each pair of sides in the diagram above:

\[
K = \frac{1}{2} ab \sin(C) = \frac{1}{2} bc \sin(A) = \frac{1}{2} ac \sin(B)
\]

If we multiply every one of these expressions by \( \frac{2}{abc} \), we get the (extended) **Law of Sines**:

\[
\frac{2K}{abc} = \frac{\sin(C)}{c} = \frac{\sin(A)}{a} = \frac{\sin(B)}{b}
\]

Another thing we can try with this triangle diagram is applying the pythagorean theorem, since \( h \) creates two right triangles. We have:

\[
b^2 = (CD)^2 + h^2, \quad c^2 = (a - CD)^2 + h^2 = a^2 - 2a(CD) + (CD)^2 + h^2
\]

Subtracting the first equation from the second to get eliminate \( (CD)^2 + h^2 \), we get the **Law of Cosines**:

\[
c^2 - b^2 = a^2 - 2a(CD) = a^2 - 2ab \cos(C) \implies c^2 = a^2 + b^2 - 2ab \cos(C)
\]
4.1 Examples

1. In \( \triangle ABC \), we have \( AB = 13 \), \( BC = 14 \), and \( AC = 15 \). Point \( P \) lies on \( BC \), and \( AP \perp BC \). What is the length of \( BP \)?

2. In \( \triangle ABC \), we have \( AB = 13 \), \( \angle A = 75^\circ \), and \( \angle B = 45^\circ \). What are the perimeter and area of \( \triangle ABC \)? (Hint: \( \sin(75^\circ) = \frac{\sqrt{2} + \sqrt{6}}{4} \))

4.2 Exercises

1. In \( \triangle ABC \), \( \angle B = 3\angle C \). If \( AB = 10 \) and \( AC = 15 \), compute the length of \( BC \).

2. (2019 AMC 12A #19) In \( \triangle ABC \) with integer side lengths, \( \cos A = \frac{11}{16} \), \( \cos B = \frac{7}{8} \), and \( \cos C = -\frac{1}{4} \). What is the least possible perimeter for \( \triangle ABC \)?

3. (2001 AIME #4) In triangle \( ABC \), angles \( A \) and \( B \) measure 60\(^\circ\) and 45\(^\circ\), respectively. The bisector of angle \( A \) intersects \( BC \) at \( T \), and \( AT = 24 \). The area of triangle \( ABC \) can be written in the form \( a + b\sqrt{c} \), where \( a, b, \) and \( c \) are positive integers, and \( c \) is not divisible by the square of any prime. Find \( a + b + c \).

4. (2017 AMC 10B #19) Let \( ABC \) be an equilateral triangle. Extend side \( AB \) beyond \( B \) to a point \( B' \) so that \( BB' = 3 \cdot AB \). Similarly, extend side \( BC \) beyond \( C \) to a point \( C' \) so that \( CC' = 3 \cdot BC \), and extend side \( CA \) beyond \( A \) to a point \( A' \) so that \( AA' = 3 \cdot CA \). What is the ratio of the area of \( \triangle A'B'C' \) to the area of \( \triangle ABC \)?