10-23-22 AMC8 Training - Geometry

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1 Introduction

Today, we'll be going over another highly present topic throughout all of the AMC tests, whether it be 8, 10, 12, or even AIME and USAMO: Geometry. It might seem daunting at first, all the images and pictures, but the concepts themselves are rooted in pretty much everything surrounding us in the world. So let's get to it, starting with one very important question...

1.1 What is Geometry?

Geometry is the study of figures in space in two dimensions, three dimensions, and even some highly advanced math in the fourth dimension (we, however, will not be doing this). It looks at the relations between, points, lines, surfaces, and everything in between. Such an open-ended area of study gives rise to all different kinds of problems surrounding its properties, and we highly recommend doing further study beyond to explore the application of what this handout has taught you!

2 Background

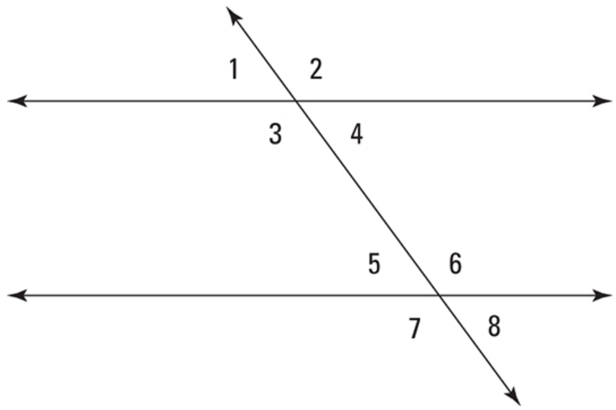
Let's first go over some background on lines and angles in geometry.

2.1 Congruence

The concept of *congruence* is that two figures that are congruent are the exact same in both size and shape. Think of it as the "equals" of a shape, angle, or any other figure–instead of numbers. So when two triangles are congruent, the triangles are exactly the same in shape and size.

The symbol that is normally used to signify congruence is $\cong$.

2.2 Lines and Angles



When two lines intersect, four angles are formed.
Angles 1 and 2 in the figure above are called *linear angles*. Linear angles are **supplementary**, meaning the sum of their measures is 180 degrees. Angles 1 and 4 above are called *vertical angles*. Vertical angles are congruent.

Parallel Lines, like the ones in the picture above are in the same plane but never intersect. When a line intersects a pair of parallel lines, though, it is called a *transversal*. Three types of angles are created by transversal lines, as seen above:

Angles 1 and 5 are Corresponding Angles (CA).

(1) Name three other pairs of corresponding angles.

Angles 5 and 4 are Alternate Interior Angles (AIA). They are called interior because they are between the parallel lines, and alternate because they are on opposite sides of the transversal. Angles 3 and 6 are also alternate interior angles.

(2) Name one other pair of alternate interior angles.

Angles 1 and 8 are Alternate Exterior Angles (AEA).

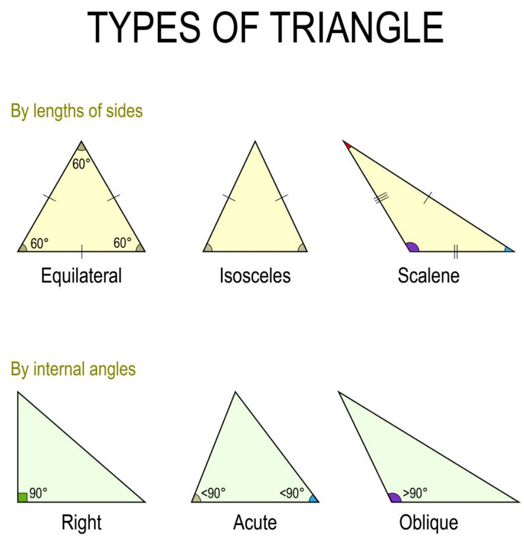
Exterior because they are outside the parallel lines, and alternate because they are on opposite sides of the transversal. Angles 2 and 7 are also alternate exterior angles.

(3) Name one other pair of alternate exterior angles.

All three of these types of angles above form congruent pairs.

3 Triangles

3.1 Triangle Basics



Triangles are classified by their sides and their angles, as seen above.

A triangle with three congruent sides is called **equilateral**. A triangle with two or more congruent sides is called **isosceles**. A triangle with no congruent sides is called **scalene**.

A triangle in which all of the angles are less than 90° is called **acute**. A triangle in which one angle is 90° is called a **right** triangle. A triangle in which one angle measure is greater than 90° is called **oblique** or **obtuse**.

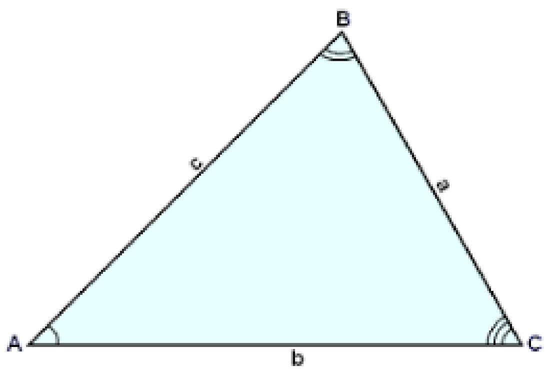
There are also two circumstances where side lengths and angles can be related. In an equilateral triangle, all of the angle measures are congruent (60°). An isosceles triangle has two congruent base angles which are opposite the congruent sides.

3.2 Triangle Angle and Side Properties

In any triangle, the largest angle will always be opposite the longest side, and the smallest angle will be opposite the shortest side. The marks seen above are used to indicate congruent sides and a right angle.

Triangle Angle Sum: We can use what we know about parallel lines to prove that the sum of the angles in a triangle is 180 degrees. **Try writing down your thoughts on how to prove that this is true!**

The Triangle Inequality: There is a property of triangles which seems almost too obvious to mention, but many competition problems have use problems that cause students to overlooked the following:



The sum of the length of any two sides of a triangle must be greater than the length of the third side:

$a + b > c$

Consequently, the difference between the lengths of any two sides of a triangle must be less than the length of the third side:

$a > c - b$

Let's try using these properties in a problem!

(1) What is the area of a triangle of integral side lengths and perimeter of 8cm?

3.3 The Pythagorean Theorem

Perhaps the most useful (and most often used) tool in Geometry is the Pythagorean theorem. The sum of the squares of the legs *of a right triangle* is equal to the square of the hypotenuse, or, as you probably know it:

$a^2 + b^2 = c^2$

Let's prove this theorem on the board!

Knowing the most common Pythagorean triples is very helpful in recognizing and solving problems. These "common ones" often include *3-4-5*, *5-12-13*, *7-24-25*, *9-40-41*, *8-15-17*, and *20-21-29*. Remember these, they come up in a lot of problems!

There are some patterns which also make many of the Pythagorean triples easy to remember. Of course, the easiest triples to remember are multiples of the common ones like 3-4-5, 6-8-10, and 15-20-25.

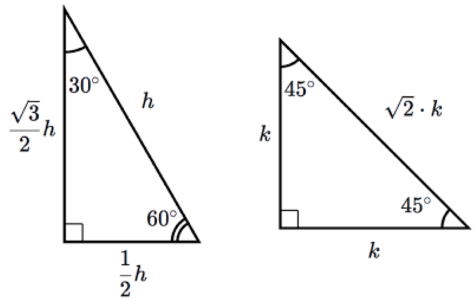
3.4 Special Right Triangles

There are two right triangles which occur quite frequently. Knowing the relationship between their sides is key to solving many competition problems which involve the Pythagorean theorem. But first, let's try solving these problems:

(1) What is the diagonal measure of a square whose sides measure 3cm?

(2) What is the altitude of an equilateral triangle whose sides measure 4cm?

So what exactly are the special right triangles?



The 45-45-90 right triangle (or one-half of a square):
Given leg length a, the hypotenuse is $a\sqrt{2}$.

The 30-60-90 right triangle (or one-half of an equilateral triangle):
The hypotenuse of a 30-60-90 right triangle is twice the length of the short leg. The long leg of a 30-60-90 right triangle is equal to the product of the short leg and $\sqrt{3}$.

3.5 Triangle Area

It is really important to know the area of a triangle! The most common formula that many of you may know already is

$A = \frac{1}{2}b * h$

where A is the area, b is the base, and h is the height of the triangle. However, there is one other formula that is also really helpful on the AMC exam! It is called *Heron's Formula*, and the entire formula is based only off of SIDE LENGTHS. It is as follows:

$A = \sqrt{(s)(s-a)(s-b)(s-c)}$

where A is the area of the triangle, a, b, and c are the side lengths of the triangle, and s is the *semi-perimeter* of the triangle. The semi-perimeter of a triangle is defined as

$s = \frac{a+b+c}{2}$

3.6 Similar Triangles

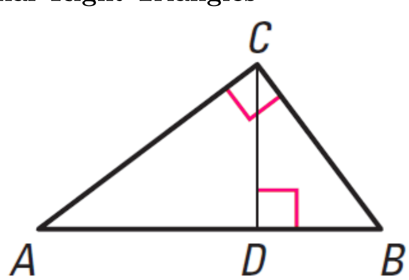
Polygons are described as similar if all corresponding angles are congruent, and all corresponding sides are proportional. In simplest terms, similar polygons are the same shape but not necessarily the same size. Polygons may be enlarged, rotated, or even reflected and remain similar to the original.

Similarity is also indicated by the symbol $\sim$, as in $ABC \sim DEF$.

Most similarity problems involve triangles. Any two triangles which share the same angle measures are similar (AAA), or if they share one angle and have two adjacent proportional sides (SAS).

The ratio of the length of the sides of similar figures is called **scale factor**. Multiplying all the sides of one figure by the scale factor will achieve all the sides of another figure, and the opposite is true with division.

Similar Right Triangles



The altitude to the hypotenuse of a right triangle divides it into two smaller right triangles which are similar to the original. Given any two of the five segments in the right triangles above, all of the remaining lengths can be found using a combination of similarities and the Pythagorean theorem. Let's try it out!

(1) The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments of length 6cm and 10cm. Find the area of the right triangle.

4 Polygons

Next, we'll be looking at shapes with more than just three sides: polygons!

4.1 Polygons Basics

What is a polygon?

A polygon is a closed plane figure. Diagonals in a polygon connect non-adjacent pairs of vertices. Polygons may be convex or concave. In a convex polygon, all of the diagonals are contained entirely within the polygon. Or, in other words, all the angle measures are less than 180° in a convex polygon. A polygon that is not convex is concave.

There are also special polygons. An **equilateral** polygon has all sides of equal length. An **equiangular** polygon has (you guessed it) all angles of equal measure. A regular polygon is equilateral and equiangular, but an equilateral or equiangular polygon is not always necessarily both.

Polygon Angle Sum We have shown that the sum of the angle measures in a triangle is 180°. We can use this to find a way to determine the sum of the angle measures in any polygon:

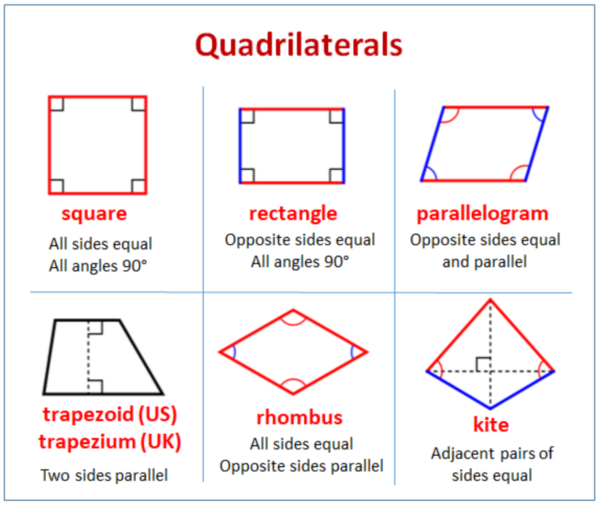
If a polygon has n sides, we can always divide it into $(n-2)$ triangles by drawing $(n-3)$ diagonals. The sum of the angle measures in each triangle is 180 degrees, so the sum of the angles in a polygon with n sides is $180(n-2)$.

Perhaps even more useful is the sum of the exterior angles in a polygon. In any polygon, the sum of its exterior angles will always equal 360°.

4.2 Polygon Types

Polygons with different numbers of side lengths have different names. A three sided polygon, as we saw, is called a triangle. A four sided polygon is called a quadrilateral. A five sided polygon is called a pentagon. Other notable polygons are:
Hexagon (6)
Heptagon or Septagon (7)
Octagon (8)
Nonagon (9)
Decagon (10) Dodecagon (12)

There are also a couple quadrilaterals that are worth noting for the AMC.



Trapezoids have exactly one pair of parallel sides, called its bases. A midsegment of a trapezoid connects the midpoints of the non-parallel sides. The midsegment will always be parallel to the bases, and its length is the average of the bases. An isosceles trapezoid has congruent non-parallel sides and congruent base angles.

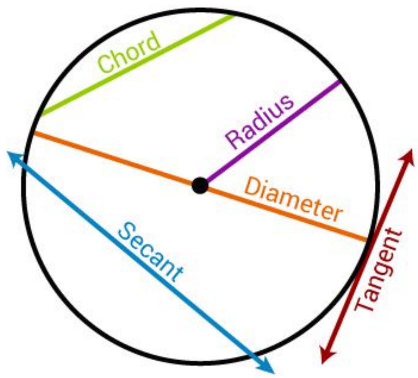
Parallelograms have two pairs of parallel sides. The parallel sides are congruent. Diagonals of a parallelogram bisect each other. If a parallelogram is equilateral, it is called a rhombus. The diagonals of a rhombus have the additional property that they are perpendicular to each other. If a parallelogram is equiangular (90 degrees for each angle) it is called a rectangle. A regular quadrilateral is a square.

A kite is a quadrilateral with to distinct pairs of congruent sides which are adjacent. The angles where congruent sides meet are called vertex angles. The angles where non-congruent sides meet are called non-vertex angles. Non-vertex angles are congruent. The diagonals of a kite are perpendicular. The diagonal connecting the vertex angles bisects the other diagonal as well as both vertex angles.

5 Circles

Now, on to circles! We can first start with some vocabulary (there is a lot!).

5.1 Circles Basics

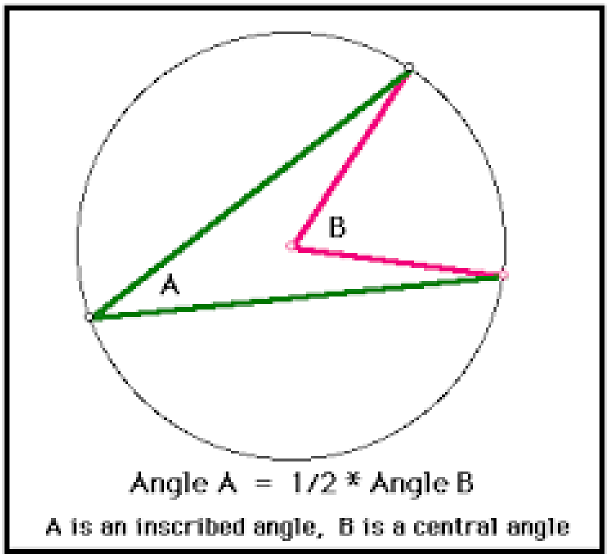


Aside from the more commonly known parts of a circle (radius, circumference, and diameter), you should also know the terms chord, secant, and tangent. As with many terms in geometry, these are easiest to define with a diagram:

The green line segment is a **chord**. The endpoints of a chord lie on the circle. A **diameter** (the orange line) is a chord which passes through the center of a circle.

The blue line is a **secant**. A secant is a line which passes through the circle.

The red line is a **tangent**. A tangent is a line which intersects a circle at exactly one point. Note that tangent can also be used as an adjective, describing two figures which touch at one point.



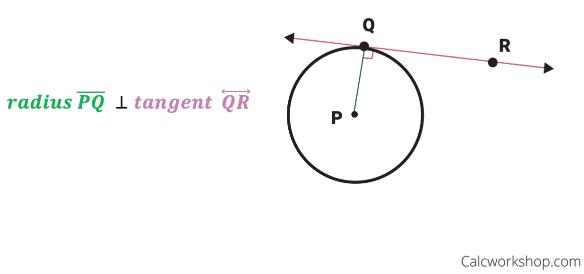
An **arc** is a portion of a circle's circumference. Arcs are measured in degrees and the measure of an arc is the same as the corresponding central angle (we will go over these more in a following section!)

An inscribed angle is an angle with its vertex and endpoints on the circumference of a circle. In the above diagram, angle A is an inscribed angle, while the angle corresponding to its arc length is angle B.

5.2 Circle Properties

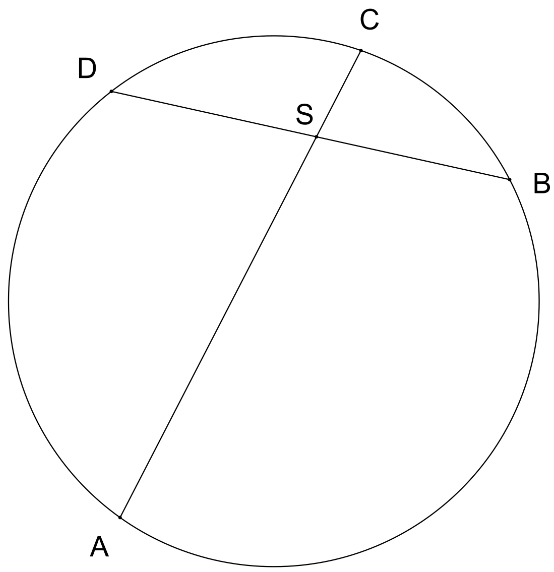
There are a couple highly notable circle properties we'll be covering today. If you're interested, you can also check out other properties as well, such as power of a point!

1) A tangent will be perpendicular to the radius drawn to the point of tangency.



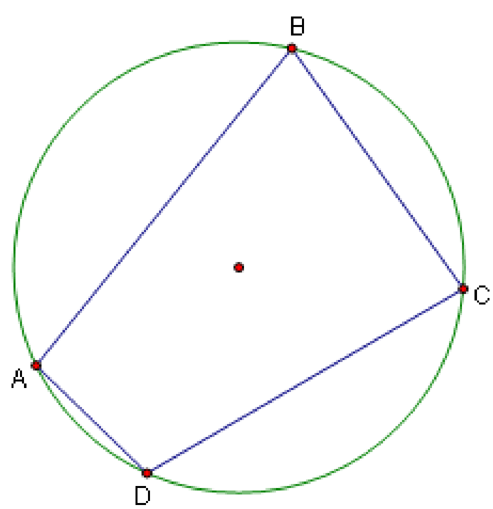
2) The measure of an inscribed angle is equal to half the measure of the intercepted arc.
See the picture from the section above!

3) The products of the line segments of created by intersecting chords are equal.



So in this graphic, $(SA)(SC) = (SD)(SB)$.

4) Opposite angles in cyclic quadrilaterals, or quadrilaterals inscribed in a circle, are supplementary: or in other words, they add up to 180.



5) The circumference of a circle is $\pi * d$ or $2\pi * r$, and the arc length of a specific angle is the ratio of that angle measure to 360 times the circumference.
Or in other words, $\theta/360 * C$ where θ is the angle measure of the arc and c is the circumference of the circle.

Let's do a problem to practice all of what we have learned here:
(1) What is the perimeter of the GREEN outline below which consists of three congruent tangent circles of radius 6cm?