OLGA RADKO MATH CIRCLE: ADVANCED 3

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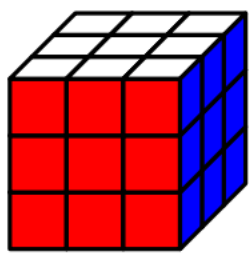
Worksheet 3: Groups and the Rubik's cube

In 1974, Ernö Rubik invented the popular three dimensional combination puzzle known as the Rubik's Cube. The cube was first launched to the public in May of 1980 and quickly gained popularity. Since its launch, 350 million cubes have been sold, becoming one of the best selling puzzles. By 1982, the cube had become part of the Oxford English Dictionary. The rubik cube has more than 43 quintillon possible configurations. In what follows, we will put the white face upward as shown in the picture:



We will fix the following notation: The upward face will be white. The front face will be red. The bottom face is yellow. The right face is blue. The left face is green. The back face is orange.

So far in this class we have learnt about groups. If we fix a initial position (solved cube) and we fix a bottom face (yellow) One can think about the Rubik's cube as a group. Any variation of the cube can be described as a sequence of rotations of its faces. We use the following notation for the rotations:

- U is the 90 degrees rotation clockwise of the upward (top) face,
- D is the 90 degrees rotation clockwise of the downwad (bottom) face,
- F is the 90 degrees rotation clockwise of the front face,
- B is the 90 degrees rotation clockwise of the back face,
- R is the 90 degrees rotation clockwise of the right face,
- L is the 90 degrees rotation clockwise of the left face.

In case that we want to rotate the rubik's cube in the counter-clockwise direction, we will use the "inverse" notation from group theory. Thus, we write U^{-1} for the 90 degrees rotation counter-clockwise of the upward (top) face. We do it similarly for the other faces, i.e., we write $D^{-1}, F^{-1}, B^{-1}, R^{-1}$, and L^{-1} for the counter-clockwise rotations. When we write a concatenation of rotations, for instance, LUF we apply them from right to left.

Problem 2.0: Take your cube an apply the FLU to it. Describe the upward face that you obtain.
Solution 2.0:

Any position of the cube can be written in terms of U, D, F, B, R, and L. Hence, understanding how to solve the Rubik's cube is analogous to understanding the group generated by these elements. We will write $\mathcal{G}_{\text{rubik}}$ for the group generated by U, D, F, B, R, and L. It is clear that $U^4 = 1$. Then, there is some presentation of $\mathcal{G}_{\text{rubik}}$ of the form:

$$\langle U, D, F, B, R, L \mid U^4 = \cdots = L^4 = 1, \dots \rangle.$$

These are some of the relations, but of course, this is not an exhaustive list. Finding a new relation in the Rubik's cube presentation, is the same than finding a sequence of steps that brings back to the "initial" or "solved" cube.

Problem 2.1: Find two other new relations of the group $\mathcal{G}_{\text{rubik}}$ that are not products of the previous listed relations. For instance, the relation $L^4R^4 = 1$ is not considered new.
Solution 2.1:

The presentation of the Rubik's cube group has around 43 relations and the length of these relations add up to 597 terms.

Definition 1. *A group G is called commutative if for every two elements $g_1, g_2 \in G$ the relation $g_1 g_2 = g_2 g_1$ holds in G. In other words, a commutative group is one for which the order of the multiplication does not matter.*

Problem 2.2: Is the group $\mathcal{G}_{\text{rubik}}$ commutative?
Consider the element $DR^{-1}L^{-1}U$ in the group $\mathcal{G}_{\text{rubik}}$. What would be the inverse operation of this element? **Solution 2.2:**

Problem 2.3: Find the least number of moves that you need to make so that the upward (top) face has at least one tile of each color.
Describe the rotations that you performed in the Rubik's cube group.
How many different ways can you find to achieve this?
Solution 2.3:

Problem 2.4: We have seen that the Rubik's cube group is generated by the elements U, D, F, B, R, L. One can wonder whether this set of generators is "minimal", or maybe there is a redundant element. This means that maybe one of the elements can be written in terms of the others.

- Show that D^2 can be written in terms of U, F, B, R, L.
- Show that D can be written in terms of U, F, B, R, L

Solution 2.4:

Problem 2.5: The group $\mathcal{G}_{\text{rubik}}$ can be generated by the two elements

$$L^2BRD^{-1}L^{-1} \qquad \text{and} \qquad UFRUR^{-1}U^{-1}F^{-1}.$$

This means that every Rubik's cube can be solved using only these two movements. However, in practice, solving the Rubik's cube with only two moves may be quite hard, as the number of relations in the group $\mathcal{G}_{\text{rubik}}$ increases a lot when we only consider two generators. Let's put this in practice. Shuffle your cube and try to solve it only using the two previous movements. Record the cube you tried to solve and the combination of the previous movements that you chose.
Solution 2.5:

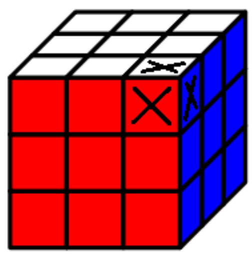
Problem 2.6: Is there a single combination of movements of the Rubik's cube such that every Rubik's cube can be solved applying this movement over and over?
Solution 2.6:

Definition 2. Let $(G, *)$ be a group with identity element 1_G. A *subgroup* S of G is a subset $S \subset G$ satisfying the following conditions:

- S is closed under $*$, meaning that for every $s_1, s_2 \in S$, we have that $s_1 * s_2 \in S$;
- the identity element 1_G is in S; and
- for every element $s \in S$, we have that the inverse s^{-1} is again in S.

A subgroup can be though as a subset which is also a group with the binary operation $*$.
For instance, the group $(\mathbb{Q}, +)$ has a subgroup $(\mathbb{Z}, +)$.

Problem 2.7: Let $\mathcal{G}_{\text{rubik}}$ be the Rubik's cube group. Consider the set of all the elements that preserves the bottom left corner of the upward face. This means that, after applying this transformation, the three marked tiles in this picture have the same colors:



Show that this set of transformations is a subgroup of $\mathcal{G}_{\text{rubik}}$.
Find a set of generators for this subgroup.
Solution 2.7:

Problem 2.8: Let $\mathcal{G}_{\text{rubik}}$ be the Rubik's cube group. Consider the set of all the elements that preserves all the white tiles in the top face. Show that this set of transformations is a subgroup of $\mathcal{G}_{\text{rubik}}$.
Find a set of generators for this subgroup.
Solution 2.8:

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