# ORMC AMC Group: Week 5
# Counting and Probability

October 23, 2022

## 1 Stars-and-Bars (Review)

### 1.1 Examples Solutions

1. If we have 10 (identical) balls, how many ways can we place them into a red bag, a green bag, a yellow bag, and a blue bag?

   We have 10 items and 4 containers, so we will represent this with 10 stars and 3 bars. A few possible distributions are:

   $$**||****|**** \qquad |*********|*| \qquad **|**|****|**$$

   The number of stars before the first bar represents the number of balls in the red bag, the number of stars between the first and second bars represents the number of balls in the green bag, and so on. In total, we have 13 spaces for 10 stars and 3 bars, which means we have $\binom{13}{3} = \boxed{286}$ ways to distribute the balls among the bags.

2. What if we want at least 1 ball in each bag? At least 2?

   If we want at least 1 ball in each bag, we start by allocating 1 ball to each bag, leaving us with 6 remaining balls. We then do stars-and-bars to distribute the 6 remaining balls (6 stars, 3 bars), in $\boxed{\binom{9}{3} = 84}$ ways.

   Similarly, if we want at least 2 in each bag, we give 2 to each bag and do stars-and-bars with the remaining 2. We have 2 stars and 3 bars, giving us $\binom{5}{2} = \boxed{10}$ total ways of distributing the balls.

3. What if we have an additional 15 balls of some other color?

   Notice that the balls of different colors are completely independent. So we can first distribute the 10 balls of the first color (same as above, $\binom{13}{3}$ ways). Then, we can distribute the 15 balls of the other color: we have 15 stars and 3 bars, giving us a total of $\binom{18}{3}$ ways.

   We then multiply these two values together, we get a total of $\binom{13}{3}\binom{18}{3} = 233376$ ways.

### 1.2 Exercises Solutions

1. How many 4 digit numbers are there such that the thousands digit is the sum of the other 3 digits?

   The thousands digit may be any number 1 through 9. If it is $k$, then we may represent the remaining 3 digits using stars-and-bars with $k$ stars and 2 bars. So, we have a total of

   $$\sum_{k=1}^{9} \binom{k+2}{2} = \sum_{k=0}^{9} \binom{k+2}{2} - \binom{2}{2}.$$

   The sum is equivalent to $\binom{10}{3}$ by the hockey-stick theorem, so our final answer is $\binom{10}{3} - \binom{2}{2} = 120 - 1 = \boxed{119}$.

2. At Peter's school, to progress to the next year, he has to pass an exam every summer. Every exam is out of 50 and the pass mark is always 25. To graduate from school, he must pass 10 exams. Since Peter's work ethic increases with age, his scores never decrease.

   Let $N$ be the number of different series of marks Peter could have achieved, given that he left school without ever failing an exam. What are the last 3 digits of $N$?

   Let each score, 25 through 50, be a "box". Let each test be a "ball", which can be placed in one of the boxes. Since the tests must be ordered in strictly increasing order, we can consider them to be "indistinguishable". So, we have a stars-and-bars problem with 26 boxes (25 bars) and 10 stars. This gives us a total of $\binom{35}{10} = 183579396$. So, the last 3 digits are $\boxed{396}$.

# 2 (More) Binomial Coefficients

## 2.1 Examples Solutions

1. **(2020 AIME I #7)** A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let $N$ be the number of such committees that can be formed. Find the sum of the prime numbers that divide $N$.

   If we pick $k$ men, then we must have $k+1$ women, where $k$ can be any value from 0 to 11. So, we can represent this as the sum:

   $$\sum_{k=0}^{11} \binom{11}{k}\binom{12}{k+1}$$

   Note that $\binom{12}{k+1} = \binom{12}{11-k}$, so we can turn the sum into the vandermonde identity:

   $$\sum_{k=0}^{11} \binom{11}{k}\binom{12}{11-k} = \binom{23}{11}$$

   By writing out the terms and doing some cancellation, we find that the prime factorization is $2 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23$. The sum of these values is $2+7+13+17+19+23 = \boxed{81}$.

2. **(1992 AIME, #4)** In Pascal's Triangle, each entry is the sum of the two entries above it. In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3:4:5$?

   WLOG, let the entries be $\binom{n}{k-1}, \binom{n}{k}, \binom{n}{k+1}$. Then, we have $4\binom{n}{k-1} = 3\binom{n}{k}$ and $5\binom{n}{k} = 4\binom{n}{k+1}$. Note that $\binom{n}{k}/\binom{n}{k-1} = \frac{n-k+1}{k}$ and $\binom{n}{k+1}/\binom{n}{k} = \frac{n-k}{k+1}$. This gives us

   $$\frac{4}{3} = \frac{n-k+1}{k} \implies 4k = 3n - 3k + 3, \qquad \frac{5}{4} = \frac{n-k}{k+1} \implies 5k+5 = 4n - 4k$$

   Solving, we get $n = 62$, which means that the three entries occur in row $\boxed{63}$.

## 2.2 Exercises Solutions

1. **(2021 AMC 12A #15)** A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let $N$ be the number of different groups that could be selected. What is the remainder when $N$ is divided by 100?

   We can separate this into 4 cases:

   (a) 8 more basses than tenors: $\binom{6}{0}\binom{8}{8}$
   (b) 4 more basses than tenors: $\sum_{k=0}^{4} \binom{6}{k}\binom{8}{k+4}$
   (c) same number of basses and tenors: $\sum_{k=0}^{6} \binom{6}{k}\binom{8}{k}$
   (d) 4 more tenors than basses: $\sum_{k=0}^{2} \binom{6}{k+4}\binom{8}{k}$

   Note that for each of these, we may use the vandermonde identity by changing each $\binom{6}{x}$ to $\binom{6}{6-x}$, or $\binom{8}{x}$ to $\binom{8}{8-x}$. So our sums become:

   (a) $\binom{6}{0}\binom{8}{8} = 1$
   (b) $\sum_{k=0}^{4} \binom{6}{k}\binom{8}{4-k} = \binom{14}{4}$
   (c) $\sum_{k=0}^{6} \binom{6}{6-k}\binom{8}{k} = \binom{14}{6}$
   (d) $\sum_{k=0}^{2} \binom{6}{2-k}\binom{8}{k} = \binom{14}{2}$

Notice that in the third case, we are counting the case where both the number of tenors and basses is 0, which is not allowed. So we need to subtract 1 from this case, which cancels out with the +1 from the first case. So, in total, we have $\binom{14}{2} + \binom{14}{4} + \binom{14}{6}$. Each of these, $\mod 100$, is $91, 1, 3$, which means the total remainder when divided by 100 is $\boxed{95}$.

2. **(2007 AIME II #13)** A triangular array of squares has one square in the first row, two in the second, and in general, $k$ squares in the $k$th row for $1 \le k \le 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below (illustrated in given diagram). In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?

   Label each of the bottom squares as $x_0, x_1 \ldots x_9, x_{10}$.

   You can try this on smaller pyramids; note that if $x_k = 1$ and all the other bottom squares are 0, then the top square will be $\binom{10}{k}$.

   So, in general, the top square will be $\binom{10}{0}x_0 + \binom{10}{1}x_1 + \cdots + \binom{10}{10}x_{10}$.

   Note that $\binom{10}{0} \equiv \binom{10}{1} \equiv \binom{10}{9} \equiv \binom{10}{10} \equiv 1 \pmod 3$. It is also not difficult to reason that since 9 shows up in the numerator of $\binom{10}{2}$, we will always have an excess 3 while computing each of $\binom{10}{2}$ through $\binom{10}{8}$, so all of these will be congruent to $0 \pmod 3$.

   This means that each of $x_2, \ldots, x_8$ can be 0 or 1 with no effect, giving us a total of $2^{8-2+1} = 2^7$ ways to set these values.

   However, we must have exactly 0 or 3 of $x_0, x_1, x_9, x_{10}$ being 1, while all the others are 0. We can do this in exactly $1 + \binom{4}{3} = 1 + 4 = 5$ ways.

   So the total number of ways that the top square will be divisible by 3 is $5 \cdot 2^7 = 10 \cdot 2^6 = \boxed{640}$.

# 3 Inclusion-Exclusion

## 3.1 Examples Solutions

1. Alice, Bob, and Charlie are in a class of 20 people. How many ways can we form a study group of at least 2 people, including at least one of Alice, Bob, and Charlie?

   In order to make a group that must contain Alice, we have $2^{19}$ possible choices (each other person can be in the group, or not– 2 choices). However, we must have at least one person other than Alice in the group, so we have to subtract the case where all the other people of the class are excluded from the group. So, there are $2^{19} - 1$ possible groups that contain Alice. Likewise, there are $2^{19} - 1$ groups that contain Bob, and $2^{19} - 1$ that contain Charlie.

   In order to make a group that must contain Alice and Bob, we have $2^{18}$ possible choices. Since a group only needs to have at least 2 people, we don't need to subtract anything here. So the total number of groups that contain both Alice and Bob is $2^{18}$. The number of groups that contain both Alice and Charlie is the same, as is the number of groups that contain both Bob and Charlie.

   Finally, there are $2^17$ groups that contain all 3 of them. So, applying the principle of inclusion-exclusion, we have a total of

   $3(2^{19}-1)-3(2^{18})+2^{17} = 2^{17}(12-6+1)-3 = 7\cdot 2^{17}-3$ groups that contain at least one of them.

## 3.2 Exercises Solutions

1. **(2017 AMC 10B #13)** There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

   Let $Y$ be the set of students in yoga, $B$ be the set of students in bridge, and $P$ the set of students in painting. Then, since the total number of students is 20, we know by the principle of inclusion-exclusion that

   $$|Y| + |B| + |P| - |Y \cap B| - |Y \cap P| - |B \cap P| + |Y \cap B \cap P| = 20$$

   Since the total number of students in at least two classes is 9, we have

   $$|Y \cap B| + |Y \cap P| + |B \cap P| - 2|Y \cap B \cap P| = 9.$$

   We also know that $|Y| + |B| + |P| = 10 + 13 + 9 = 32$. Add the two top equations and subtract the bottom one to get:

   $$-|Y \cap B \cap P| = -3$$

   So the number of students in all 3 classes is $\boxed{3}$.

2. **(2005 AMC 12A #18)** Call a number prime-looking if it is composite but not divisible by $2, 3$, or 5. The three smallest prime-looking numbers are $49, 77$, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

   The given states that there are 168 prime numbers less than 1000, which is a fact we must somehow utilize. Since there seems to be no easy way to directly calculate the number of "prime-looking" numbers, we can apply complementary counting. We can split the numbers from 1 to 1000 into several groups: $\{1\}$, $\{\text{numbers divisible by } 2 = \mathrm{S}_2\}$, $\{\text{numbers divisible by } 3 = \mathrm{S}_3\}$, $\{\text{numbers divisible by } 5 = \mathrm{S}_5\}$, $\{\text{primes not including } 2,3,5\}$, $\{\text{prime} - \text{looking}\}$. Hence, the number of prime-looking numbers is $1000 - (168 - 3) - 1 - |S_2 \cup S_3 \cup S_5|$ (note that $2, 3, 5$ are primes).

   We can calculate $S_2 \cup S_3 \cup S_5$ using the Principle of Inclusion-Exclusion:

   $$\begin{aligned} |S_2 \cup S_3 \cup S_5| &= |S_2| + |S_3| + |S_5| - |S_2 \cap S_3| - |S_3 \cap S_5| - |S_2 \cap S_5| + |S_2 \cap S_3 \cap S_5| \\ &= 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734 \end{aligned}$$

   Substituting, we find that our answer is $1000 - 165 - 1 - 734 = \boxed{100}$.

3. **(2021 AMC 10B #22)** Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. The probability that at least one box receives 3 blocks all of the same color is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m+n$?

   Let our denominator be $(5!)^3$, so we consider all possible distributions.

   We use PIE (Principle of Inclusion and Exclusion) to count the successful ones.

   When we have at least 1 box with all 3 balls the same color in that box, there are $\binom{5}{1}\cdot\binom{5}{1}\cdot 1!\cdot(4!)^3$ ways for the distributions to occur: $\binom{5}{1}$ for selecting one of the five boxes for a uniform color, $\binom{5}{1}$ for choosing the color for that box, 1! for ordering the color among that (one) box, and 4! for each of the three people to place their remaining items.

   However, we overcounted those distributions where two boxes had uniform color, and there are $\binom{5}{2}\cdot\binom{5}{2}\cdot 2!\cdot(3!)^3$ ways for the distributions to occur: $\binom{5}{2}$ for selecting two of the five boxes for a uniform color, $\binom{5}{2}$ for choosing the color for those boxes, 2! for ordering the colors among those boxes, and 3! for each of the three people to place their remaining items.

   Again, we need to re-add back in the distributions with three boxes of uniform color... and so on so forth.

   By the principle of inclusion-exclusion, we end up with

$$\binom{5}{1}\cdot 1!\cdot(4!)^3-\binom{5}{2}\cdot 2!\cdot(3!)^3+\binom{5}{3}\cdot 3!\cdot(2!)^3-\binom{5}{4}\cdot 4!\cdot(1!)^3+\binom{5}{5}^2\cdot 5!\cdot(0!)^3=120\cdot 2556.$$

$$\frac{120\cdot 2556}{120^3}=\frac{71}{400},$$

   yielding an answer of $\boxed{471}$.

4. How many 12-character passwords can you make if you must include at least one digit, one special character, and one letter? Assume there are 8 special characters, 10 digits, and 26 letters. What if you must also include a capital letter?

   We do this by complementary counting, with inclusion-exclusion. Let $A$ be the set of passwords that include no digits. Let $B$ be the set of passwords that contain no special characters. Let $C$ be the set of passwords that contain no letters.

   Then, if $\Omega$ is the set of all passwords, our desired set of passwords is $\Omega\setminus(A\cup B\cup C)$.

   ($A\setminus B$ indicates that we want all elements which are in $A$, but not in $B$.)

   As usual, we can calculate $|A\cup B\cup C|$ using the principle of inclusion-exclusion.

$$|\Omega|=(8+10+26)^{12},\quad |A|=(8+26)^{12},\quad |B|=(10+26)^{12},\quad |C|=(8+10)^{12}$$

$$|A\cap B|=26^{12},\quad |B\cap C|=10^{12},\quad |C\cap A|=8^{12}$$

$$|A\cap B\cap C|=0$$

$$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$$

   Our final answer is $|\Omega|-|A\cup B\cup C|\approx 4.56\cdot 10^{19}$ (no need to actually calculate this).

   If we want to also include capital letters, we add a set $D$ which consists of all passwords not containing any capital letters. This will clearly be larger than the last answer, so again, no need to calculate this. But to see how we would work it out, the set sizes are:

$$|\Omega|=(8+10+26+26)^{12},\quad |A|=(8+26+26)^{12},\quad |B|=(10+26+26)^{12},$$

$$|C|=(8+10+26)^{12},\quad |D|=(8+10+26)^{12}$$

$$|A\cap B|=(26+26)^{12},\quad |B\cap C|=(10+26)^{12},\quad |C\cap A|=(8+26)^{12},$$

$$|A \cap D| = (8+26)^{12}, \quad |B \cap D| = (10+26)^{12}, \quad |C \cap D| = (8+10)^{12}$$
$$|A \cap B \cap C| = 26^{12}, \quad |A \cap B \cap D| = 26^{12}, \quad |A \cap C \cap D| = 8^{12}, \quad |B \cap C \cap D| = 10^{12}$$
$$|A \cap B \cap C \cap D| = 0$$

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$-|A \cap B| - |A \cap C| - |B \cap C| - |A \cap D| - |B \cap D| - |C \cap D|$$
$$+|A \cap B \cap C| + |A \cap C \cap D| + |A \cap B \cap D| + |B \cap C \cap D|$$

And as before, our final answer is given by $|\Omega| - |A \cup B \cup C \cup D|$.

# 4 Geometric Probability

## 4.1 Examples Solutions

1. You are throwing darts at a circular dartboard. You never miss, but each dart lands randomly on the dartboard. What is the probability that a dart lands closer to the center than to the edge of the dartboard?

   WLOG, let the dartboard have radius $R$. Then, the dart lands closer to the center precisely when it is in an inner circle of radius $R/2$. Since the darts land randomly, the probability that we end up in this inner circle is:

   $$\frac{\text{Area of inner circle}}{\text{Area of outer circle}} = \frac{\pi(R/2)^2}{\pi R^2} = (1/2)^2 = \boxed{\frac{1}{4}}$$

2. Your bus comes to the stop at a random time between 12 pm and 1 pm. If you show up at 12:30, what is the probability that you catch the bus?

   You will catch the bus precisely if the bus shows up at 12:30 or later. Since the bus shows up randomly between 12 and 1, we may model this as a random point on the interval $[0, 1]$. Then, the bus showing up after 12:30 corresponds to the points on the interval $[0.5, 1]$. So, the probability that you catch the bus ends up being:

   $$\frac{\text{length of } [0.5, 1]}{\text{length of } [0, 1]} = \frac{0.5}{1} = \boxed{\frac{1}{2}}.$$

## 4.2 Exercises Solutions

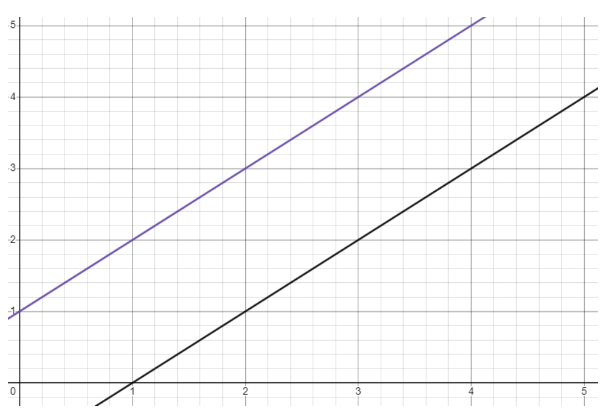
1. Alice and Bob each arrive at a coffee shop at some random time between 8 A.M. and 1 P.M., and each of them stays for exactly an hour before leaving. What is the probability that they see each other?

   We model the time period from 8 am to 1 pm as the interval $[0, 5]$, where 0 corresponds to 8 am, 1 corresponds to 9 am, and so on. Then, let the x-axis be Alice's arrival time, and let the y-axis be Bob's arrival time. Each combination of arrival times corresponds to some point in the square given by $0 \leq x \leq 5$ and $0 \leq y \leq 5$.

   If Alice shows up first, Bob must show up at most 1 hour later. We can represent this situation by the inequality $y \leq x + 1$.

   Similarly, if Bob shows up first, Alice must show up at most 1 hour later. We can represent this situation by the inequality $x \leq y + 1$.

   Combining these, we find that the area we want should be within the given square between these two lines:




   We can find that this area is 9, and the total area of the square is $5^2 = 25$, so the probability that they see each other will be $\frac{9}{25}$.

2. **(2017 AMC 12A #10)** Chloe chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chloe's number?

   Let $x$ be the number chosen randomly by Chloe. Because it is given that the number Chloe chooses is in the interval $[0, 2017]$, $0 \leq x \leq 2017$. Next, let $y$ be the number chosen randomly by Laurent. Because it is given that the number Laurent chooses is in the interval $[0, 4034]$, $0 \leq y \leq 4034$. Since we are looking for when Laurent's number is greater than Chloe's we write the equation $y > x$. When these three inequalities are graphed the area captured by $0 \leq x \leq 2017$ and $0 \leq y \leq 4034$ represents all the possibilities, forming a rectangle 2017 in width and 4034 in height. Thus making its area $4034 * 2017$. The area captured by $0 \leq x \leq 2017$, $0 \leq y \leq 4034$, and $y > x$ represents the possibilities of Laurent winning, forming a trapezoid with a height 2017 in length and bases 4034 and 2017 length, thus making an area $2017 * \frac{4034+2017}{2}$. The simplified quotient of these two areas is the probability Laurent's number is larger than Chloe's, which is $\boxed{\frac{3}{4}}$.

3. Three points are placed at random on the circumference of a circle. What is the probability that they form an acute triangle?

   As we will discuss later when we cover geometry, the three points will form an acute triangle precisely when there is no diameter that places all three points to one side. So, it suffices to find the probability that all three points are on the same half of the circle, and subtract that from 1.

   Note that any 2 points can be contained in the same half of the circle with probability 1. We can realize this by first placing one point $A$, and then placing the second point $B$. The smaller arc between them will have a measure less than 180 degrees, or it will have a measure precisely equal to 180 degrees. Precise equality essentially never happens, because there are infinitely many possibilities for the angle between the two, and only one gives an arc measure of 180, so this has probability 0. Hence, we can say that any 2 points are in the same half of the circle with probability 1.

   Now, let the smaller arc between the first two points have measure $0 < \alpha < 180$. If it is to be in the same half as the first two points, the third point $C$ can be:

   (a) along the arc (which has measure $\alpha$.)
   (b) outside the arc and close enough to $A$ that it lies in the same half. Note that the measure of arc $BAC$ must be at most 180, so the length of the arc $AC$ is at most $180 - \alpha$.
   (c) outside the arc and close enough to $B$ that it lies in the same half. Note that the measure of arc $ABC$ must be at most 180, so the length of the arc $BC$ is at most $180 - \alpha$.

   Adding these up, for a given placement of $A$ and $B$, the total measure of where $C$ can be placed in order to lie in the same half as both of them is $360 - \alpha$. Recall that $\alpha$ ranges from 0 to 180. So if we graph it on this domain, the area we are interested occurs under the line. It is easy to see that this area takes up 3/4 of the total area, so the three points are on the same side with probability 3/4.

   That is, they are not on the same side with probability 1/4, so there is a $\boxed{1/4}$ probability that they form an acute triangle.

4. **(2018 AMC 10B #22)** Real numbers $x$ and $y$ are chosen independently and uniformly at random from the interval $[0, 1]$. Which of the following numbers is closest to the probability that $x, y$, and 1 are the side lengths of an obtuse triangle?

   As we will discuss in later lessons, it turns out that in an obtuse triangle, $a^2 + b^2 < c^2$, where $c$ is the longest side. The triangle inequality tells us that $a + b > c$, where $c$ is again the longest side. So, we have two inequalities:

$$x^2 + y^2 < 1$$

$$x + y > 1$$

   The first equation is $\frac{1}{4}$ of a circle with radius 1, and the second equation is a line from $(0, 1)$ to $(1, 0)$. So, the area is $\frac{\pi}{4} - \frac{1}{2}$ (which is approximately 0.29).

5. **(2015 AMC 10A #25)** Let $S$ be a square of side length 1. Two points are chosen independently at random on the sides of $S$. The probability that the straight-line distance between the points is at least $\dfrac{1}{2}$ is $\dfrac{a-b\pi}{c}$, where $a$, $b$, and $c$ are positive integers with $\gcd(a,b,c)=1$. What is $a+b+c$?

   Let one point be chosen on a fixed side. Then the probability that the second point is chosen on the same side is $\frac{1}{4}$, on an adjacent side is $\frac{1}{2}$, and on the opposite side is $\frac{1}{4}$. We discuss these three cases.

   Case 1: Two points are on the same side. Let the first point be $a$ and the second point be $b$ in the $x$-axis with $0 \le a, b \le 1$. Consider $(a,b)$ a point on the unit square $[0,1]\times[0,1]$ on the Cartesian plane. The region $\{(a,b) : |b-a| > \frac{1}{2}\}$ has the area of $(\frac{1}{2})^2$. Therefore, the probability that $|b-a| > \frac{1}{2}$ is $\frac{1}{4}$.

   Case 2: Two points are on two adjacent sides. Let the two sides be $[0,1]$ on the x-axis and $[0,1]$ on the y-axis and let one point be $(a,0)$ and the other point be $(0,b)$. Then $0 \le a, b \le 1$ and the distance between the two points is $\sqrt{a^2+b^2}$. As in Case 1, $(a,b)$ is a point on the unit square $[0,1]\times[0,1]$. The area of the region $\{(a,b) : \sqrt{a^2+b^2} \le \frac{1}{2}, 0 \le a, b \le 1\}$ is $\frac{\pi}{16}$ and the area of its complementary set inside the square (i.e. $\{(a,b) : \sqrt{a^2+b^2} > 1/2, 0 \le a, b \le 1\}$ ) is $1-\frac{\pi}{16}$. Therefore, the probability that the distance between $(a,0)$ and $(0,b)$ is at least $\frac{1}{2}$ is $1-\frac{\pi}{16}$.

   Case 3: Two points are on two opposite sides. In this case, the probability that the distance between the two points is at least $1/2$ is obviously 1.

   Thus the probability that the probability that the distance between the two points is at least $1/2$ is given by

$$\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\left(1-\frac{\pi}{16}\right)+\frac{1}{4}=\frac{26-\pi}{32}.$$

   Therefore $a=26$, $b=1$, and $c=32$. Thus, $a+b+c=\boxed{59.}$