ORMC AMC Group: Week 5
Counting and Probability

October 23, 2022

1 Stars-and-Bars (Review)

As a reminder, stars-and-bars is a technique used when we want to divide identical objects into distinct containers. We can represent this for n objects and m containers by drawing our n objects as "stars", and we can represent our m containers using $m - 1$ dividers (bars) (i.e.: $\underbrace{**\ldots*}_{n}\ \underbrace{||\ldots|}_{m-1}$).

Then, the total number of ways to distribute the n items into the m containers is the same as the number of ways we can re-order these stars-and-bars, which is $\binom{n+m-1}{n}$.

1.1 Examples

1. If we have 10 (identical) balls, how many ways can we place them into a red bag, a green bag, a yellow bag, and a blue bag?

   We have 10 items and 4 containers, so we will represent this with 10 stars and 3 bars. A few possible distributions are:

   $$**||****|**** \qquad |*********|*| \qquad **|**|****|**$$

   The number of stars before the first bar represents the number of balls in the red bag, the number of stars between the first and second bars represents the number of balls in the green bag, and so on. In total, we have 13 spaces for 10 stars and 3 bars, which means we have $\binom{13}{3} = \boxed{286}$ ways to distribute the balls among the bags.

2. What if we want at least 1 ball in each bag? At least 2?

3. What if we have an additional 15 balls of some other color?

1.2 Exercises

1. How many 4 digit numbers are there such that the thousands digit is the sum of the other 3 digits?

2. At Peter's school, to progress to the next year, he has to pass an exam every summer. Every exam is out of 50 and the pass mark is always 25. To garduate from school, he must pass 10 exams. Since Peter's work ethic increases with age, his scores never decrease.

   Let N be the number of different series of marks Peter could have achieved, given that he left school without ever failing an exam. What are the last 3 digits of N?

2 (More) Binomial Coefficients

A few additional facts about the binomial coefficients which are useful to know:

1. Recurrence Relation:
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
This is easy to see by expanding the coefficients.

2. Vandermonde (Convolution) Identity:
$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$$
To understand this, consider the following. Suppose we have n A's and m B's, and we want to choose k letters. We may choose i A's and $k-i$ B's, for all $0 \le i \le k$. Or, we may simply choose k out of our total $n+m$ letters. One special case of this is when $m = k = n$, which gives us:
$$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

2.1 Examples

1. **(2020 AIME I #7)** A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

2. **(1992 AIME, #4)** In Pascal's Triangle, each entry is the sum of the two entries above it. In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3 : 4 : 5$?

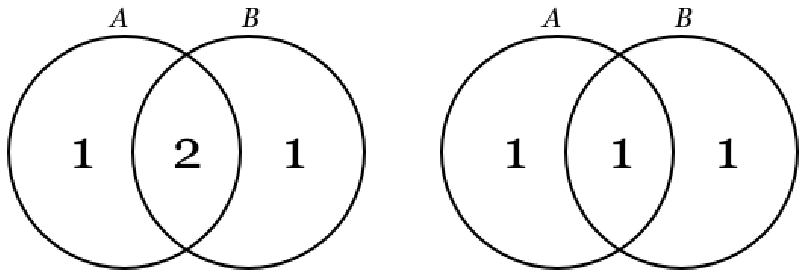
2.2 Exercises

1. **(2021 AMC 12A #15)** A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of different groups that could be selected. What is the remainder when N is divided by 100?

2. **(2007 AIME II #13)** A triangular array of squares has one square in the first row, two in the second, and in general, k squares in the kth row for $1 \le k \le 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below (illustrated in given diagram). In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?

3 Inclusion-Exclusion

You probably are already familiar with the concept of "over-counting", where we count two things that have an overlap, and then subtract their overlap. This can be represented with a venn diagram:

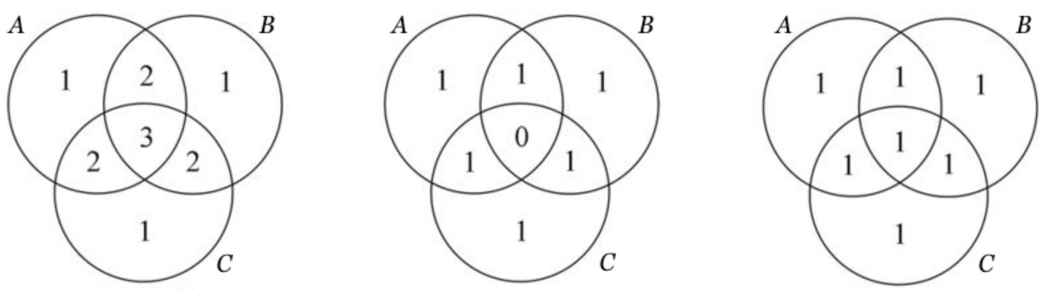


$|A| + |B|$ $\qquad$ $|A| + |B| - |A \cap B|$

The **Principle of Inclusion-Exclusion** tells us how to generalize this to an arbitrary number of different events. If we have n different events $A_1, A_2, \ldots, A_n$, and event A_i can happen in $|A_i|$ different ways, then the number of ways for at least one of the events to happen is:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

When we had only 2 events, we added their totals, then subtracted out the (2-event) overlap. The formula above tells us that in general, we add all the totals, subtract the 2-event overlaps, add back the 3-event overlaps, subtract the 4-event overlaps, and so on. For example, when we have 3 events, a venn-diagram representation might look like the following:



$|A| + |B| + |C|$

$|A| + |B| + |C|$
$- |A \cap B| - |A \cap C| - |B \cap C|$

$|A| + |B| + |C|$
$- |A \cap B| - |A \cap C| - |B \cap C|$
$+ |A \cap B \cap C|$

3.1 Examples

1. Alice, Bob, and Charlie are in a class of 20 people. How many ways can we form a study group of at least 2 people, including at least one of Alice, Bob, and Charlie?

3.2 Exercises

1. **(2017 AMC 10B #13)** There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

2. **(2005 AMC 12A #18)** Call a number prime-looking if it is composite but not divisible by $2, 3$, or 5. The three smallest prime-looking numbers are $49, 77$, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

3. **(2021 AMC 10B #22)** Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. The probability that at least one box receives 3 blocks all of the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

4. How many 12-character passwords can you make if you must include at least one digit, one special character, and one letter? Assume there are 8 special characters, 10 digits, and 26 letters. What if you must also include a capital letter?

4 Geometric Probability

Geometric probability is a method of calculating probability by using length, area, or volume. When outcomes are random, the probability of something happening is the area it corresponds to, divided by the total area. Many probability questions which do not explicitly involve geometry, can be simplified by reframing them as a geometric probability question.

4.1 Examples

1. You are throwing darts at a circular dartboard. You never miss, but each dart lands randomly on the dartboard. What is the probability that a dart lands closer to the center than to the edge of the dartboard?

2. Your bus comes to the stop at a random time between 12 pm and 1 pm. If you show up at 12:30, what is the probability that you catch the bus?

4.2 Exercises

1. Alice and Bob each arrive at a coffee shop at some random time between 8 A.M. and 1 P.M., and each of them stays for exactly an hour before leaving. What is the probability that they see each other?

2. **(2017 AMC 12A #10)** Chloe chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chloe's number?

3. Three points are placed at random on the circumference of a circle. What is the probability that they form an acute triangle?

4. **(2018 AMC 10B #22)** Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. Which of the following numbers is closest to the probability that x, y, and 1 are the side lengths of an obtuse triangle?

5. **(2015 AMC 10A #25)** Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a - b\pi}{c}$, where a, b, and c are positive integers with $\gcd(a, b, c) = 1$. What is $a + b + c$?