Geometry of Graphs I

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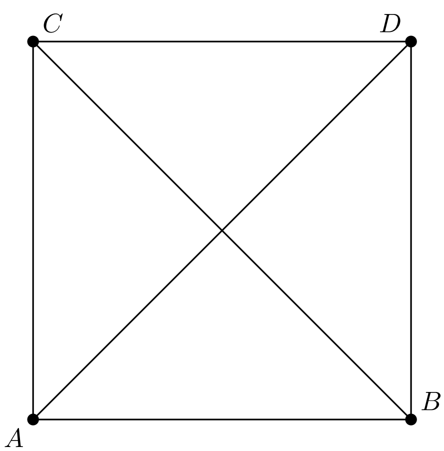
ORMC Advanced 1 - Fall 2022

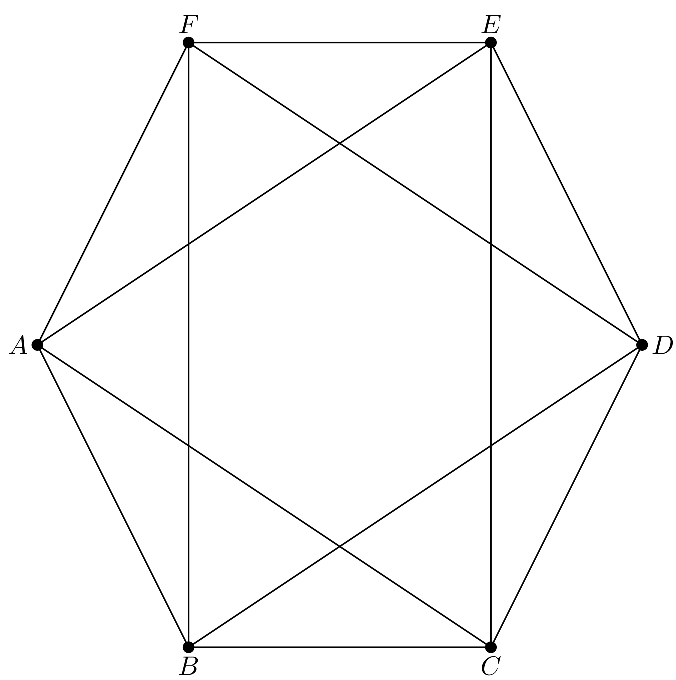
1 Planar Graphs

Definition 1 *A* ***graph*** G *consists of a set of vertices* $V(G)$ *and edges* $E(G)$ *between pairs of vertices.*

We usually require a few extra properties of our graphs. As a reminder, they are *connectedness*, which is the property that any two vertices are connected by an edge or sequence of edges (alternatively, that it's possible to travel along the graph between any two vertices), and *simpleness*, which is the property that there is no edge from any vertex to itself, nor more than one edge between any pair of vertices.

(Connected, simple) Graphs can be drawn in the plane, by drawing points corresponding to the vertices and line segments (or curves!) connecting vertices that are connected by an edge. The following diagrams represent connected simple graphs.

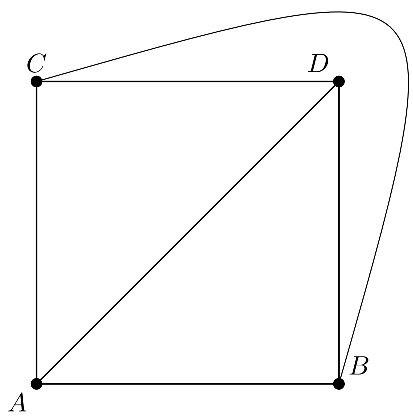


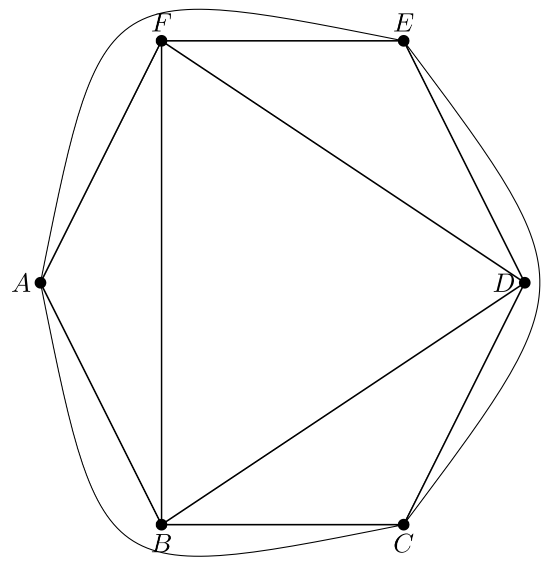


Almost all examples of graphs that occur in mathematics and in applications are connected and simple, and from now on we will assume that all of our graphs are connected and simple. In these worksheets, we will also focus on a special type of graphs called *planar graphs*.

Definition 2 *A (connected, simple) graph G is* ***planar*** *if it can be drawn in the plane without edges intersecting (except at vertices).*

Problem 1 *The two example graphs on the previous page are actually both planar! Prove this by redrawing them without intersecting edges.*





Solution:

Unlike connectedness and simpleness, planarity is not universal in common examples of graphs. Fortunately, we recall a trick that allows us to check nonplanar graphs.

Definition 3 *Given a planar graph G drawn without intersecting edges, the regions that the edges of G divide the plane into are called the* ***faces*** *of G.*

In addition to faces bounded by edges (which are called *interior faces*), every graph also has one *exterior face*, which can be thought of a face "at infinity", which contains everything that's outside the graph.

Definition 4 *Given a graph G with V vertices, E edges, and F faces, its* ***Euler characteristic*** *is given by $\chi(G) = V - E + F$.*

Theorem 1 *(Euler) The Euler characteristic of every planar graph is 2.*

(A good exercise is to check this on both of the graphs you drew in Problem 1.)

The proof of this theorem is outlined in many different Math Circle worksheets (including at least one from last year), so let us only sketch it as a reminder:

1. First, we show that the formula holds for *trees*, or (connected, simple) graphs with only one face, by induction.
2. Given any planar graph G, we then find a *spanning tree*, which is a tree containing every vertex of G that only uses the edges of G. (We can use one of several algorithms to find such a tree.)
3. Given a spanning tree of G, we then add all the other edges of G, showing that the formula remains true at every step.

Problem 2 *(Bonus) Complete the proof of Theorem 1 by filling in the details of each step.*

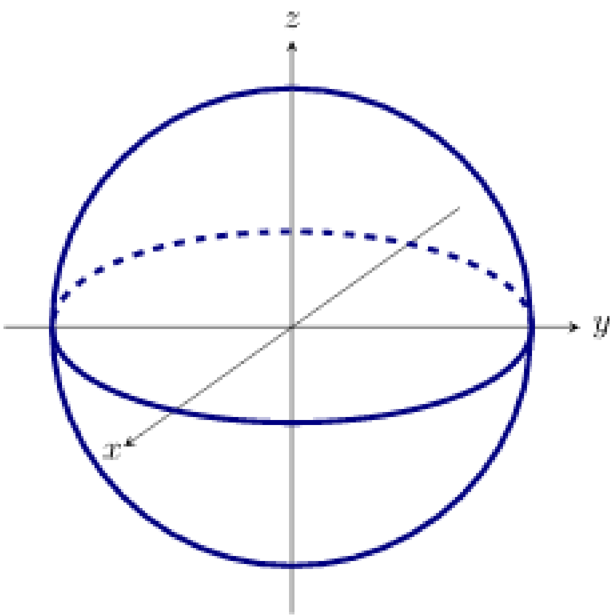
2 The Sphere

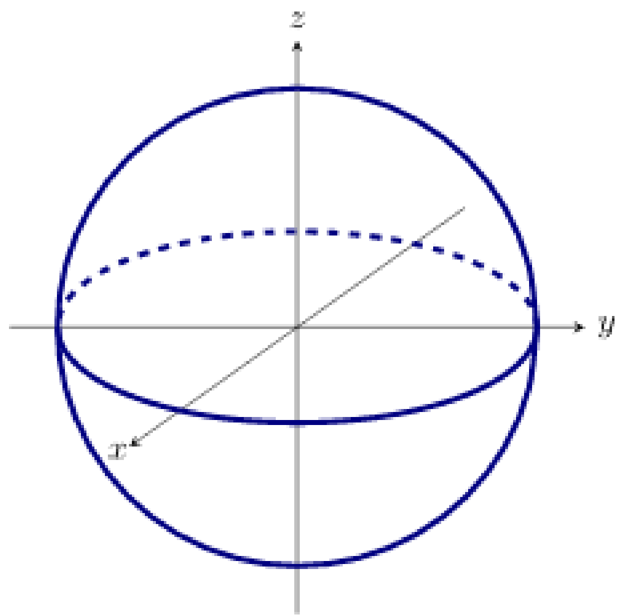
We define the 2-sphere (denoted S^2) as the set of points in 3-dimensional Euclidean space that are distance 1 away from the origin. More precisely,

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

(Note that we could also define spheres S^n of different dimensions the same way inside $(n+1)$-dimensional space. We won't be working with anything other than S^2, though.)

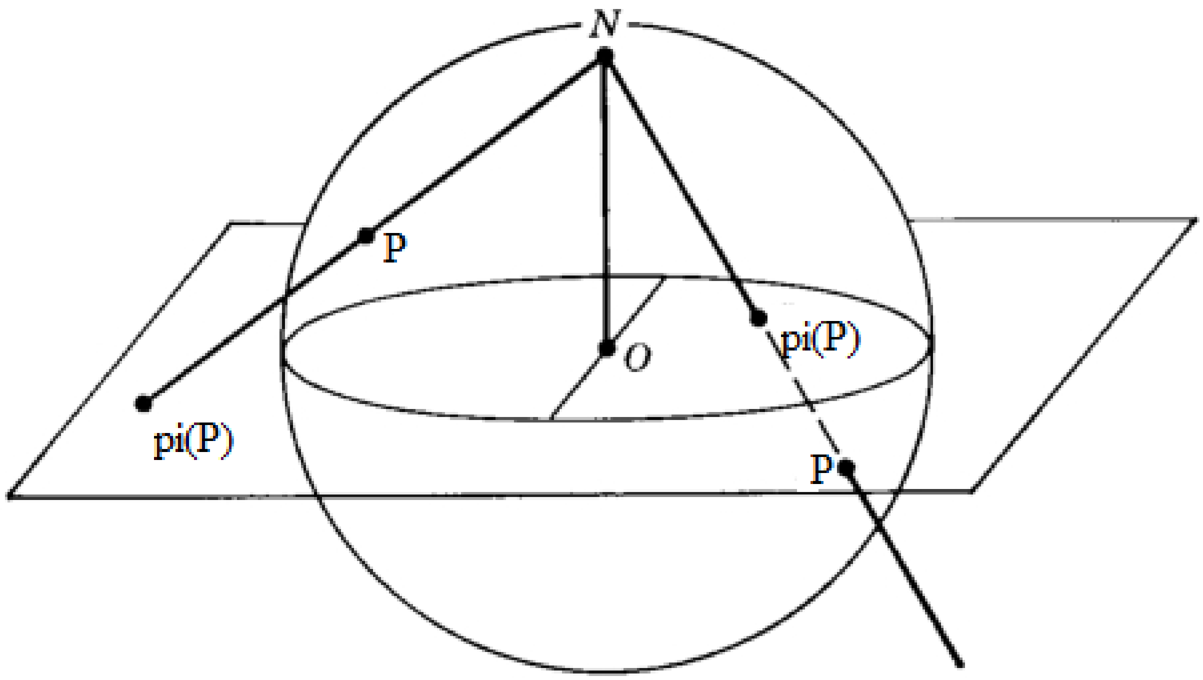
Problem 3 *For each graph in Problem 1, determine if it can also be drawn on S^2 without crossing edges. (Copies of S^2 have been provided below for attempts.)*





Solution: It is possible for both.

S^2 is related to the plane $\mathbb{R}^2$ by the following function, known as *stereographic projection*: fix a north pole N of S^2 (we typically choose $N = (0, 0, 1)$). For any point $P \neq N$ on S^2, let $\pi(P)$ be the x and y coordinates of the point where the plane $z = 0$ intersects the line NP. In other words, $\pi : S^2 - \{N\} \to \mathbb{R}^2$. This is illustrated below:



(For a different north pole, we rotate the entire picture and can redefine the function as such.)

Problem 4 *Find the images of the following geometric objects on S^2 under π:*

- *A circle not passing through N.*
- *A circle passing through N.*
- *Two circles tangent to each other at N.*

Solutions: The first is a circle, the second is a line, and the third is two parallel lines.

Problem 5 *Show that $\pi : S^2 - \{N\} \to \mathbb{R}^2$ is a bijection (that is, that π is one-to-one and onto).*

Solution: (One-to-one) Two different points on the sphere give two different lines which intersect at N, so they must intersect the plane at two different points, since different lines only intersect once. (Onto) Given any point on the plane, there is a line from that point to N, which must intersect the sphere somewhere.

Problem 6 *Fix the standard north pole $N = (0,0,1)$. Given any point $(x,y,z) \in S^2$ except the north pole, find a formula for $\pi(x,y,z)$.*

Solution: We intersect the line $(0,0,1)+t(x,y,z-1)$ with the plane $z=0$ (alternatively, use similar triangles) to obtain

$$\pi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

From Problem 6, we see that as P gets closer to the north pole, its z-coordinate gets closer to 1, so $pi(P)$ gets larger and larger. As a result, we can define

$$\pi(N) := "\infty"$$

By Problem 5, we see that there is a bijection $S^n \to \mathbb{R}^2 \cup \{\infty\}$, where ∞ is a "point at infinity" (importantly, only one point at infinity!) This leads to the following classical identification, and the way we will think of spheres from now on:

$$S^2 \simeq \mathbb{R}^2 \cup \{\infty\}$$

where we think of the north pole (and therefore any point on S^2 by rotational symmetry) as ∞.

Problem 7 *Prove that every planar graph can be drawn on S^2 without crossing edges, and has the same number of faces on S^2 that it does on the plane.*

Solution: A planar graph can already be drawn this way on the plane, so just add ∞ to the same drawing. Since the face at infinity is already a face, this doesn't change the number of faces.

Problem 8 *Prove that every graph that can be drawn on S^2 without crossing edges is planar. (Hint: Vertices and edges have zero area, so just find a good place to put ∞.)*

Solution: The vertices and edges of such a graph have zero area while the sphere has positive surface area, so there exists a point of S^2 not lying on a vertex or edge. Letting that point be ∞, we just stereographically project the entire graph.

3 Graphs and Polyhedra

Definition 5 *A **polyhedron** is a solid figure in three-dimensional space* $\mathbb{R}^3$ *with flat polygonal faces, straight edges, and vertices. A polyhedron is called a **Platonic solid** if it is convex, all of its faces are congruent regular polygons, and the same number of faces meet at every vertex.*

Geometers have studied Platonic solids specifically since ancient Greek civilization. We'll study them with a more modern approach, by viewing them as graphs.

Problem 9 *Show that every convex polyhedron can be viewed as a planar graph with the same number of vertices, edges, and faces. (Hint: A convex polyhedron can be deformed into a sphere. Make sure the edges don't cross, then use Problem* 8*.)*

Solution: Don't require that much formality, just saying something along the lines of "puffing out all the edges doesn't make them cross" is fine.

We can apply theorems we've proven about planar graphs to polyhedra, which gives us useful properties, and even allows us to classify all possible Platonic solids.

Problem 10 *Prove that for any polyhedron,* $E \leq 3V - 6$*. (Hint: Any polyhedron has to have at least four vertices. For any planar graph with at least four vertices, consider the minimum edges per face, and plug that into the Euler characteristic equation.)*

Solution: For planar graphs with at least 3 vertices, each face has at least three edges. Since each edge is shared by exactly two faces, $2E \leq 3F$. Rearranging gives the desired $E \leq 3V - 6$.

Problem 11 *Let us classify all Platonic solids.*

- Let p be the number of edges of each face and q be the number of faces meeting at each vertex. Why is q also equal to the number of edges meeting at each vertex?

  Solution: Every face meets a vertex with two edges, and every edge borders two faces.

- Show that

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$$

  Solution: Since every edge has two faces and two vertices on it, we see that $F = 2E/q$ and $V = 2E/p$. Plugging this into Euler's characteristic equation and rearranging gives

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2}$$

- Why are p and q both at least 3? Use this to show that p and q are both at most 5.

  Solution: p is at least three because the polygon with the minimum number of edges is a triangle, and q is at least 3 because with 2 edges, the figure cannot possibly be three-dimensional. Therefore $1/p$ and $1/q$ are at most $1/3$, so $1/p$ and $1/q$ are both greater than $1/6$, so p and q are both less than 6.

- Find all possible solutions $\{p, q\}$, and therefore all possible Platonic solids.

  Solution: $\{3, 3\}, \{3, 4\}, \{3, 5\}, \{4, 3\}, \{5, 3\}$

- (Bonus) Draw the graph of each solution you've found.