COMBINATIONS I

MATH CIRCLE (INTERMEDIATE) 5/6/2012

Recall that the number of ways to choose a group of k objects from a set of n objects is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ (read "n choose k").

1) Suppose a teacher needs to bring cookies for her class of 15 students.

a) How many ways can she hand out the cookies if she has 6 chocolate chip cookies, 5 oatmeal raisin cookies, and 4 sugar cookies?

b) Repeat a) if she has 5 of each cookie type.

2) Suppose you are dealt 5 cards (in no specific order) from a standard deck of cards.

a) How many ways are there to get all cards of the same suit?

b) How many ways are there to get exactly one pair?

c) How many ways are there to get exactly two pairs?

d) How many ways are there to get a three of a kind and a pair?

e) How many ways are there to get all the cards in order, that is
$A, 2, 3, 4, 5; 2, 3, 4, 5, 6; \ldots; 10, J, Q, K, A$?

3) Suppose you start at $(0, 0)$ in the (x, y)-plane. The only valid moves are 1 unit in the $+x$ direction or 1 unit in the $+y$ direction. (So from $(0, 0)$ you can move to either $(1, 0)$ or $(0, 1)$.)

a) Where can you get in exactly three moves?

b) How many ways are there to get to each of the places in a) ?

c) How many ways are there to get to the point $(4, 3)$?

d) How many ways are there to get to the point (m, n)?

4) Prove the following "combinatorially":

a) $\binom{n}{k} = \binom{n}{n-k}$.

b) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

c) $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$. Hint: Think about subsets of $\{1, 2, 3, \ldots, n\}$.

Challenge 1) Prove combinatorially that $\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$, that is $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$. Hint: Problem 3 might help.

Challenge 2) Prove combinatorially that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$, that is $\sum_{k=0}^{n} (-1)^n \binom{n}{k} = 0$. Hint: First rewrite the equation without any subtractions.

Challenge 3) Prove by induction that $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F(n+1)$, where $F(0) = 0, F(1) = 1, F(n) = F(n-1) + F(n-2)$. Hint: Use strong induction, and deal with the cases where n is even or odd separately.

Problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"
- Previous UCLA Math Circle notes