# ORMC AMC Group: Week 4
# Combinatorics

October 16, 2022

## 1 Permutations Solutions

1. Find the number of 4-letter words you can make from the word HELLO.

   There are two cases:

   - **L appears 1 time**: The remaining letters are H,E,L,O, so there are 4! possibilities in this case.
   - **L appears 2 times**: There are 2 distinct letters X,Y$\in \{H, E, O\}$, and 2 L's. So, there are 4!/2! possibilities in this case. The important thing to notice is that this case happens 3 times: for each of H, E, or L being excluded. So there are actually a total of $3 \cdot 4!/2!$ possibilities in this case

   In total, we have $4!(1 + 3/2) = 24 \cdot 5/2 = \boxed{60}$ possible words total.

2. Let $S$ be the set of words formed by re-arranging the word "ANGLE" arranged in alphabetical order. Which place would the word "GLEAN" be in? (The letters arranged in alphabetical order is $\{A, E, G, L, N\}$.)

   We figure this out by counting how many words come before "GLEAN" alphabetically. There are:

   (a) All the words starting with "A" or "E"
   (b) All the words starting with "GA" or "GE"
   (c) All the words starting with "GLA"

   We can stop here, because after "GLA", we get "GLE", and the first word that starts with "GLE" must start with "GLEA", which means that this word is "GLEAN".

   There are 4! words that start with each of "A" and "E", 3! words that start with each of "GA" and "GE", and 2! words that start with "GLA". This gives a total of $24 + 24 + 6 + 6 + 2 = 62$ words that come before "GLEAN", meaning that GLEAN is word number $\boxed{63}$.

3. Let $S$ be the set of permutations of the sequence $\{1, 2, 3, 4, 5\}$ for which the first term is not 1. A permutation is chosen randomly from $S$. Find the probability that the second term is 2.

   The key with problems like these is to always start with the tightest restriction. Let us say that $A$ is the set of all permutations where the second term is 2, and the first term is not 1.

   For $S$, we have a restriction only on the first spot, giving us only 4 choices for that spot. The rest of the numbers may be placed in any of the remaining spots, in any of the 4! possible ways. This gives us $4 \cdot 4! = 96$ total possible permutations in $S$.

   For $A$, we have restrictions on both the first and second spots, but the second spot allows only one number, so we will start with that one. We have 1 choice for the second spot (the number 2). After that, there are 3 numbers left that we can place in the first spot (3,4,5). The remaining numbers may be placed in the remaining 3 spots in any of the 3! possible ways. This gives us a total of $3 \cdot 1 \cdot 3! = 18$ possible permutations in $A$

   So, the probability that we pick an element of $A$ out of $S$ is $18/96 = \boxed{3/16}$.

## 2 Combinations Solutions

1. A 50-card deck consists of 4 cards labeled "i" for $i = 1, 2, ..., 12$ and 2 cards labeled "13". If Bob randomly chooses 2 cards from the deck without replacement, what is the probability that his 2 cards have the same label?

   For each $i$ from 1 to 12, there are 4 choose $2 = 6$ ways to select a pair of $i$s. For $i = 13$, there are 2 choose $2 = 1$ ways. So our total number of working pairs is $12 * 6 + 1 = 73$ out of $\binom{50}{2} = 1225$ pairs. This gives us a total probability of $\boxed{\frac{73}{1225}}$

2. Let $a_i = \binom{2022}{i}$. For what value of $i$ is $a_i$ the largest?

   Let us examine the transition from $a_i$ to $a_{i+1}$. To get from $a_i$ to $a_{i+1}$, we multiply by

   $$\frac{2022 - i + 1}{i} \cdot \frac{1}{2}.$$

   So, we would like to find the first value of $i$ such that $a_{i+1} < a_i$, i.e.

   $$\frac{2022 - i + 1}{i} \cdot \frac{1}{2} < 1$$

   $$2022 - i + 1 < 2i \implies 2023 < 3i \implies i > 674.$$

   This means that $a_{675} < a_{674}$, so the max value is attained at $\boxed{i = 674}$

3. The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17}$, where $y = x + 1$ and the $a_i$'s are constants. Find the value of $a_2$. (Hint: Hockey-Stick Identity)

   Rearranging gives us $x = y - 1$. We substitute for $x$:

   $$1 - (y - 1) + (y - 1)^2 + (y - 1)^3 + \cdots + (y - 1)^{16} - (y - 1)^{17}$$

   $$= 1 + (1 - y) + (1 - y)^2 + (1 - y)^3 + \cdots + (1 - y)^{16} + (1 - y)^{17}.$$

   The total number of $y^2$ powers is

   $$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{16}{2} + \binom{17}{2} = \binom{18}{3} = 816$$

   by the Hockey-stick Identity.

4. (Hard) Find a closed form expression for

   $$\binom{3n}{0} + \binom{3n}{3} + \cdots + \binom{3n}{3n}.$$

   The only way to do this, without an unhealthy amount of guessing, in addition to some induction, is to know roots of unity. If you're not familiar with roots of unity, don't worry too much. It's not a big AMC topic, and we will go over it eventually in a later lesson. Let $\omega = \cos(2\pi/3) + i\sin(2\pi/3)$, a 3rd root of unity. We have that

   $$(1 + 1)^{3n} = \sum_{k=0}^{3n} \binom{3n}{k}$$

   $$(1 + \omega)^{3n} = \sum_{k=0}^{3n} \binom{3n}{k} \omega^k$$

   $$(1 + \omega^2)^{3n} = \sum_{k=0}^{3n} \binom{3n}{k} \omega^{2k}$$

By the properties of the 3rd roots of unity, $1 + \omega^k + \omega^{2k} = 0$ wherever $k$ is not a multiple of 3. If $k$ is a multiple of 3, then we get $1 + \omega^k + \omega^{2k} = 3$. So, if we add together all the equations above, we will get:

$$(1+1)^{3n} + (1+\omega)^{3n} + (1+\omega^2)^{3n} = \sum_{k=0}^{n} 3\binom{3n}{3k}$$

So, our desired result is:

$$\frac{(1+1)^{3n} + (1+\omega)^{3n} + (1+\omega^2)^{3n}}{3}$$

After working with the values a bit, this simplifies to:

$$\frac{2^{3n} + 2(-1)^n}{3}.$$

## 3 Complementary Counting Solutions

1. **(AMC 10A 2006 #21)** How many four-digit positive integers have at least one digit that is a 2 or a 3?

   There are 7 digits to choose the leading digit from. Afterwards, there are 8 digits to choose each non-leading digit. So the total number of 4-digit numbers with no 7's is $7 \cdot 8 \cdot 8 \cdot 8$. Our answer is

   $$9 \cdot 10 \cdot 10 \cdot 10 - 7 \cdot 8 \cdot 8 \cdot 8 = 9000 - 3584 = \boxed{5416}.$$

2. Sally is drawing seven houses. She has four crayons, but she can only color any house a single color. In how many ways can she color the seven houses if at least one pair of consecutive houses must have the same color?

   First, we count the complement: how many ways we can color the houses if no two consecutive houses may have the same color. For the first house, we may choose one of 4 colors. Then, for each following house, we may choose any of the 3 colors that were not used for the previous house. So, there are $4 \cdot 3^6$ ways to color the houses in this way.

   In total, there are $4^7$ ways to color all the houses, since we may color each of them one of the 4 colors. So, our desired event can happen in $4^7 - 4 \cdot 3^6 = \boxed{13468}$ ways.

3. **(AIME I 2001 #1)** Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

   First, we count the total number of license plates: 26 choices for each letter, 10 choices for each digit, giving a total $26^3 \cdot 10^3$ license plates.

   Then, we count the total number of license plates which have no palindrome. In order for this to happen, the only requirements are that the first and last letter must not be the same, and the first and last digit must not be the same. So for the letters, we have 26 choices for the first and second, and 25 choices for the third. Similarly, we have 10 choices for the first and second digit, and 9 choices for the third. So there are $26^2 \cdot 25 \cdot 10^2 \cdot 9$ license plates with no palindrome.

   So, the probability that a license plate has at least one palindrome is

   $$1 - \frac{25 \cdot 26^2 \cdot 9 \cdot 10^2}{26^3 \cdot 10^3} = 1 - \frac{9 \cdot 25}{26 \cdot 10} = \frac{260 - 225}{260} = \frac{35}{260} = \boxed{\frac{7}{52}}$$

4. **(AMC 12B 2008 #22)** A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

   The total number of ways the cars could fill the spaces is $\binom{16}{12} = \binom{16}{4} = 4 \cdot 5 \cdot 7 \cdot 13 = 1820$.

   Then, if auntie em can find a space, some two open spots must be next to each other. We consider the complement of this event, which happens when no two open spots are next to each other. In order to count how many ways this can happen, we first place the 12 cars next to each other, and then consider how we might place the open spots between them. In order for no two spots to end up next to each other, no two of them can go between the same two cars. So, there are 13 total places that the open spots could end up (11 spaces between cars, 1 on each of the two ends). Thus, we have $\binom{13}{4} = 13 \cdot 11 \cdot 5 = 715$ ways for Auntie Em to not find a space.

   Thus, the probability that she can park is $1 - \frac{715}{1820} = \frac{1105}{1560} = \boxed{\frac{17}{28}}$.

## 4 Sticks and Stones Solutions

1. **(AMC 10 2001 #19)** Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

   There are 4 objects (stars) and 3 flavors (containers), which means we have $3 - 1 = 2$ dividers (bars). So, the total number of different selections is $\binom{4+2}{2} = \binom{6}{2} = \boxed{15}$.

2. **(AMC 10A 2003 #21), modified** Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected? What if he wants at least one of each?

   If he doesn't require at least one of each, then we can simply do 6 stars and $3 - 1 = 2$ bars, giving us a total of $\binom{8}{2} = \boxed{28}$ choices of cookies.

   If he wants at least one of each, we can first allocate one cookie to each flavor, and then distribute the remaining 3 using stars-and-bars. So, we have 3 stars and 2 bars remaining, which gives us a total of $\binom{5}{2} = \boxed{10}$ choices of cookies.

3. **(AMC 10A 2018 #11)** When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

   $$\frac{n}{6^7},$$

   where $n$ is a positive integer. What is $n$?

   We can do stars and bars, where the stars are a value of 1 and the dividers separate the dice. So, if there are 3 stars between two adjacent bars, that indicates that some die rolled a 3. Note that each die roll must have a value of at least 1. So, similar to the previous problem, we first allocate one star to each die before doing the stars and bars with the rest. This leaves us with 3 stars and 6 bars, giving us a total of $\binom{9}{3} = \boxed{84}$ ways to roll a sum of 10.

4. **(AMC 10B 2020 #25)** Let $D(n)$ denote the number of ways of writing the positive integer $n$ as a product

   $$n = f_1 \cdot f_2 \cdots f_k,$$

   where $k \geq 1$, the $f_i$ are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

   Note that $96 = 2^5 \cdot 3$. We can start by distributing the 2's with stars-and-bars. Then, once the 2's are distributed, we may place the 3 with some 2's, or between consecutive 2's, or before or after all the 2's. Note that when distributing the 2's, there must be at least one 2 in each container (between consecutive bars). The stars will be the 2's, and the bars will divide the 2's into different $f_i$'s. The bars may go in any of the 4 spaces between the 2's. So, if we have $k$ bars, there are $\binom{4}{k}$ ways to place the bars. We break up into cases depending on how many containers we are breaking up the 2's into:

   - 1 container: 0 bars $\Longrightarrow$ 1 way to distribute the 2's, 3 ways to place the 3.
   - 2 containers: 1 bars $\Longrightarrow$ 4 ways to distribute the 2's, 5 ways to place the 3.
   - 3 containers: 2 bars $\Longrightarrow$ 6 ways to distribute the 2's, 7 ways to place the 3.
   - 4 containers: 3 bars $\Longrightarrow$ 4 ways to distribute the 2's, 9 ways to place the 3.

- 5 containers: 4 bars $\Longrightarrow$ 1 way to distribute the 2's, 11 ways to place the 3.

This gives a total of $1 \cdot 3 + 4 \cdot 5 + 6 \cdot 7 + 4 \cdot 9 + 1 \cdot 11 = 3 + 20 + 42 + 36 + 11 = \boxed{112}$ different ways to make the $f_i$'s.