ORMC AMC Group: Week 4
Combinatorics

October 16, 2022

1 Modular Exponentiation Exercises

1. Find the remainder when $1 + 7 + 7^2 + \cdots + 7^{2022}$ is divided by 1000.

   Since 1000 is a large modulus, we will break it down into 8 and 125, and use these separately, before applying the chinese remainder theorem to get our final answer.

   Notice that $7 \equiv -1 \pmod 8$, so our final answer will be $1 - 1 + 1 - 1 + \cdots + (-1)^{2022} \equiv 1011 - 1011 + 1 \equiv 1 \pmod 8$.

   Then, for 125, we may use the difference-of-powers factorization to rewrite this sum as

   $$\frac{7^{2023} - 1}{7 - 1} \equiv \frac{7^{2023} - 1}{6} \equiv (7^{2023} - 1) \cdot 6^{-1} \equiv (7^{2023} - 1) \cdot 21 \pmod{125}.$$

   Notice that we couldn't do this for 8, because $\gcd(6, 8) = 2 > 1$, so we would not be able to do anything about the 6 in the denominator. By Euler's Totient theorem, we have $7^100 \equiv 1 \pmod{125}$, so $7^{2023} \equiv 7^{23} \pmod{125}$. By repeated squaring, we find that

   $$7^23 \equiv 7^{16}7^4 7^2 7^1 \equiv 101 \cdot 26 \cdot 49 \cdot 7 \equiv 1 \cdot 93 \equiv 93 \pmod{125}$$

   So our desired value is $(93 - 1)21 \equiv 57 \pmod{125}$.

   This brings us to a system of congruences:

   $$x \equiv 1 \pmod 8$$
   $$x \equiv 57 \pmod{125}$$

   Notice that $57 \equiv 1 \pmod 8$, so this must be our solution. That is, $1 + 7 + 7^2 + \cdots + 7^{2022}$ leaves a remainder of $\boxed{57}$ when divided by 1000.

2. Determine the remainder when $2004^{2003^{2002}}$ is divided by 1000. Note that 2004 is not coprime to 1000, but it is coprime to 125. And, 2004 is even and 2003^{2002} is certainly bigger than 3, so $2004^{2003^{2002}}$ is going to be divisible by 8. So we can focus on $2004^{2003^{2002}} \pmod{125}$. Euler's totient theorem tells us that $2004^{100} \equiv 1 \pmod{125}$, so we can focus on breaking down $2003^{2002} \pmod{100}$. The totient of 100 is $\phi(100) = 1 \cdot 2 \cdot 4 \cdot 5 = 40$, so:

   $$2003^{2002} \equiv 3^{2002} \equiv 3^{50 \cdot 40 + 2} \equiv (3^{40})^{50} \cdot 3^2 \equiv 1 \cdot 9 \pmod{125}$$

   Thus, we have

   $$2004^{2003^{2002}} \equiv 2004^9 \equiv 4^9 \equiv 2^{18} \pmod{125}.$$

   We know that $2^7 = 128 \equiv 3 \pmod{125}$, so we can simplify further:

   $$2^{18} \equiv 3^2 2^4 \equiv 9 \cdot 16 \equiv 144 \equiv 19 \pmod{125}.$$

   So, we now have the system of congruences:

   $$x \equiv 19 \pmod{125}$$
   $$x \equiv 0 \pmod 8$$

   Conveniently, we find that $19 \equiv 3 \pmod 8$, and $125 \equiv 5 \pmod 8$, so our solution $\bmod 1000$ is $125 + 19 = \boxed{144}$.

3. (**AMC 12A 2008 #15**) Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$? It is important to notice that what we are asking for here is $k^2 + 2^k \pmod{10}$. Since 2 is not coprime to 10, and 10 is small, it is easiest to find a pattern for 2^n. Notice that $2^1 = \equiv 2$, $2^2 \equiv 4, 2^3 \equiv 8$, $2^4 \equiv 6$, and$2^5 \equiv 2$. So for $n \geq 1$, 2^n repeats every 4 values of n. So, we should find $k \pmod{10}$ for k^2, and we should find $k \pmod 4$ for 2^k.

   Both 2008^2 and 2^{2008} are going to be divisible by 4, so k is also divisible by 4. That is, $k \equiv 0 \pmod 4$, which means $2^k \equiv 6 \pmod{10}$.

   Then, we have
$$k \equiv 2008^2 + 2^{2008} \equiv 8^2 + 2^{4 \cdot 502} \equiv 4 + 6 \equiv 0 \pmod{10}.$$
   This means $k^2 \equiv 0^2 \equiv 0 \pmod{10}$.

   So, we have $k^2 + 2^k \equiv 0 + 6 \equiv \boxed{6 \pmod{10}}$.

4. (**AMC 12A Fall 2021 #10**) The base-nine representation of the number N is $27{,}006{,}000{,}052_9$. What is the remainder when N is divided by 5? If a number has a base-9 representation $(a_n a_{n-1} \cdots a_1 a_0)_9$, we may also write this as:
$$\sum_{i=0}^{n} a_i 9^i \equiv \sum_{i=0}^{n} a_i 4^i \equiv \sum_{i=0}^{n} a_i (-1)^i \pmod 5.$$
   So, our answer will be the alternating sum of the digits, starting from the ones digit, mod 5:

   $2-5+0-0+0-0+6-0+0-7+2 = 2-5+6-7+2 = -3-1+2 = -2 \equiv \boxed{3 \pmod 5}$