ORMC AMC Group: Week 4
Combinatorics

October 16, 2022

1 Permutations

A permutation of a set is a rearrangement of its elements. The number of ways to re-order all n-objects is $n!$, where $n! = 1 \cdot 2 \cdot \dots \cdot n$. The number of ways to rearrange r out of n objects, where order matters is

$$\begin{aligned} {}_nP_r &= n \cdot (n-1) \cdot \dots \cdot (n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

1.1 Permutations with Repeats

The permutation formula works best when all your objects are different. But sometimes, you can swap two identical objects and still have the same permutation. To not overcount, if an object is repeated n times, we divide by n! for each repeated object.
If the i-th element has a_i copies, then the total number of permutations is:

$$\frac{(a_1 + a_2 + \dots + a_n)!}{a_1! a_2! \dots a_n!}.$$

1.2 Permutations Exercises

1. Find the number of 5-letter words you can make from the word THERE.

2. I have 3 identical red books and 3 identical blue books. How many different ways can I arrange them on my shelf?

1.3 Problems

1. Find the number of 4-letter words you can make from the word HELLO.

2. Let S be the set of words formed by re-arranging the word "ANGLE" arranged in alphabetical order. Which place would the word "GLEAN" be in?

3. Let S be the set of permutations of the sequence $\{1, 2, 3, 4, 5\}$ for which the first term is not 1. A permutation is chosen randomly from S. Find the probability that the second term is 2.

2 Combinations

A combination of a set is a way to choose some subset of its elements, where order doesn't matter. Recall that the number of ways to rearrange r out of n objects, where order matters, is ${}_nP_r = \dfrac{n!}{(n-r)!}$ Thus, the number of ways to choose r out of n objects, where order doesn't matter, is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

This is also known as the *binomial coefficient* and can be generated iteratively using Pascal's triangle.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

2.1 Combinations Applications and Tools

1. Pascal's Identity:
$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$
2. Hockey-Stick Identity: for integers n, r where $n \geq r$,
$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

2.2 Combinations Exercises

1. Find the number of ways to choose a committee of 5 from 8 people.
2. Find the number of ways to split the class into pairs if there are 6 people.
3. Evaluate the sum
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

2.3 Problems

1. A 50-card deck consists of 4 cards labeled "i" for $i = 1, 2, ..., 12$ and 2 cards labeled "13". If Bob randomly chooses 2 cards from the deck without replacement, what is the probability that his 2 cards have the same label?
2. Let $a_i = \binom{2022}{i}$. For what value of i is a_i the largest?

3. The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17}$, where $y = x + 1$ and the a_i's are constants. Find the value of a_2. (Hint: Hockey-Stick Identity)

4. (Hard) Find a closed form expression for

$$\binom{3n}{0} + \binom{3n}{3} + \cdots + \binom{3n}{3n}.$$

3 Complementary Counting

You may often find that it is much easier to count the elements you *don't* want. In this case, you can just count the total, and subtract the number of elements you don't want, to get your answer. This is called **complementary counting**.

3.1 Complementary Counting Exercises

1. Find the number of way to re-order the word GOODBYE such that the two O's are not adjacent.

2. How many numbers from 0 to 1000 are not multiples of 3 or 5?

3. How many 5-digit numbers have at least one 7?

3.2 Problems

1. **(AMC 10A 2006 #21)** How many four-digit positive integers have at least one digit that is a 2 or a 3?

2. Sally is drawing seven houses. She has four crayons, but she can only color any house a single color. In how many ways can she color the seven houses if at least one pair of consecutive houses must have the same color?

3. **(AIME I 2001 #1)** Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. **(AMC 12B 2008 #22)** A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

4 Sticks and Stones

Sticks and Stones (also known as “stars and bars” or “balls and urns”) is a strategy generally used to partition a set of “identical” elements into “distinct” groups.
Let’s suppose we are ordering 12 bagels, either plain or blueberry. We can buy 5 plain and 7 blueberry:

$$* * * * * | * * * * * * *$$

or 9 plain and 3 blueberry:

$$* * * * * * * * * | * * * .$$

The ”divider”, |, represents the separation of two groups. Notice that there is 1 divider, and 12 bagels, for a total of 13 positions.

In general, the number of ways to split n identical objects split among m people is equivalent to choosing $m-1$ dividers among $n+(m-1)$ positions, for a total of

$$\binom{n+m-1}{m-1}.$$

4.1 Sticks and Stones Exercises

1. Find the number of ways to distribute 10 candies to 3 children.

2. Find the number of ways to give 11 dollars to 4 people if each person must receive at least one dollar.

4.2 Problems

1. **(AMC 10 2001 #19)** Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

2. **(AMC 10A 2003 #21), modified** Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected? What if he wants at least one of each?

3. **(AMC 10A 2018 #11)** When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

   $$\frac{n}{6^7},$$

   where n is a positive integer. What is n?

4. **(AMC 10B 2020 #25)** Let $D(n)$ denote the number of ways of writing the positive integer n as a product

   $$n = f_1 \cdot f_2 \cdots f_k,$$

   where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?