ORMC Olympiad Group
Fall: Week 3
Geometry: Similarity, Triangles and Circles

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October 13, 2022

Theorem 1 (**Conditions of Being Cyclic Quadrilateral**). *There are four equivalent conditions for four points A, B, C, D, or quadrilateral $ABCD$ being cyclic*

1. *A quadrilateral $ABCD$ is cyclic if and only if $\angle ABD = \angle ACD$.*
2. *A quadrilateral is cyclic if and only if its opposite angles sum to $180°$.*
3. ***(Power of Interior Point)** Let $ABCD$ be a quadrilateral, and its diagonals AC and BD intersect at X. Then it is cyclic if and only if $AX \cdot XC = BX \cdot XD$.*
4. ***(Power of Outer point)** The opposite sides AB and CD intersect a point P. Then $ABCD$ is cyclic iff $PA \cdot PB = PC \cdot PD$.*

***Proof**: Similarity. We did this last week.*

Problems

1. **(TNMO-FR 2018 - modified)** ABC is right triangle with hypotenuse AB and it is given that $AC/BC = 3/4$. The interior circle touches sides BC and AC at D and E respectively. AD intersects with the incircle

again at the point S. Similarly BE intersects with the incircle again at T. BE and AD intersect at point K.

(a) Find AS/KD

(b) Find $(AS/TD)^2$

2. **(HMMT 2005 General)** A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

3. **(Math Prize for Girls 2014)** Let ABC be a triangle. Points D, E, and F are respectively on the sides BC, CA, and AB of ABC. Suppose that
$$\frac{AE}{AC} = \frac{CD}{CB} = \frac{BF}{BA} = x$$
for some x with $\frac{1}{2} < x < 1$. Segments AD, BE, and CF cut the triangle into 7 nonoverlapping regions: 4 triangles and 3 quadrilaterals. The total area of the 4 triangles equals the total area of the 3 quadrilaterals. Compute the value of x. Express your answer in the form $\frac{k-\sqrt{m}}{n}$, where k and n are positive integers and m is a square-free positive integer. **Remark: Note that the figure is very similar to the figure in M.2**

4. **(Prasolov 1.13)** In $\triangle ABC$ bisectors AA_1 and BB_1 are drawn. Prove that the distance from any point M of A_1B_1 to line AB is equal to the sum of distances from M to AC and BC.

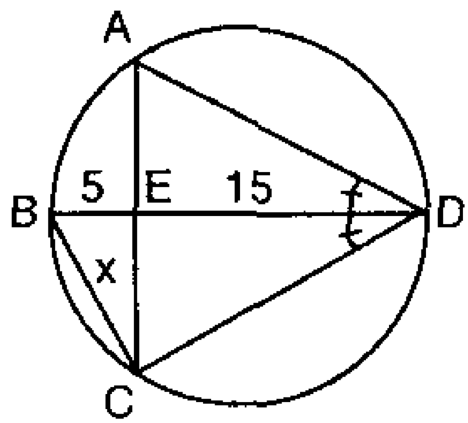
5. $ABCD$ is a parallelogram, M is the midpoint of the side AD. Let H be the feet of the altitude from C to BM. Prove that $DH = CD$.

6. **(LAMC 2008)** Prove that in any triangle a median drawn to a side is smaller than half of the sum of the other two sides.

7. From a point A outside of the circle Γ the tangent AB is drawn, where B is the tangency point. Another line which passes through A cuts the circle Γ at points C and D. If $BC = 5, BD = 7$, what the maximum integer length that the segment AB can take?

8. **(ZG)** In the figure the diameter BD is also angle bisector of $\angle ADC$.

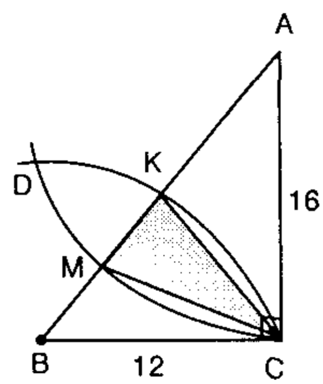
It is given $BE = 5, ED = 15$. Find $BC = x$.



9. **(ZG-modified)** In the figure the right triangle ABC has sides $BC = 12, CA = 16$. The circle with center A and radius AC cuts AB again at M, and similarly the circle with center B and radius BC cuts AB again at K.

   (i) Find the area of the triangle KMC.

   (ii) Find $\sin \angle KCM$.



10. **(ZG)** In the figure O is the center, E is the midpoint of CD, and

$AE \parallel OC$. If $OE = EC$, find $\angle BAE = \alpha$.