ORMC AMC Group: Week 2 Solutions

October 2, 2022

1 Divisibility Solutions

1. The 7-digit numbers $74A52B1$ and $326AB4C$ are each multiples of 3. What is the sum of all possible values of C?.

   Since they are both multiples of 3, we know that $7+4+A+5+2+B+1$ and $3+2+6+A+B+4+C$ are both divisible by 3. The sums are $19+A+B$ and $12+A+B+C$, so taking their difference, $7-C$ must be a multiple of 3. So, we can easily see that the only options are $1, 4, 7$, since C is a digit.

2. What is the prime factorization of 2023?

   It's not immediately clear what to do here, so we might as well try some divisibility tests for numbers we know. Clearly, it's not divisible by $2, 3, 5$, so we can check 7. We have: $202 - 2 \cdot 3 = 196$, $19 - 2 \cdot 6 = 7$. So, 2023 is divisible by 7. Dividing out, we have $2023 = 7 \cdot 289$, and we know $289 = 17^2$, and 17 is prime. So the prime factorization of 2023 is $7 \cdot 17^2$.

3. Which members of the sequence $101, 10101, 1010101, \ldots$ are prime?

   This one is much trickier. The hints given in class were:

   - Write each number as a series: $1 + 100 + 100^2 + \ldots + 100^n$.
   - Consider numbers with even and odd values of n separately.

   The first thing to notice, then, is that for odd values of n, we have an even amount of 1's in the original number, so the number will be divisible by 101, if it is greater than 101. For even values of n, we can first rewrite the series in the usual way:

   $$1 + 100 + 100^2 + \ldots + 100^n = \frac{100^{n+1} - 1}{100 - 1}$$

   And then we can apply difference-of-squares factoring to the top and bottom, since both 100 and 1 are perfect squares:

   $$= \frac{10^{n+1} - 1}{10 - 1} \cdot \frac{10^{n+1} + 1}{10 + 1}$$

   Now, it's very important that n is even, so that we can write it as $2m$ for some integer m. This is because we need to apply the difference- and sum-of-powers factoring identities from last week's worksheet.

   Remember, we have $a^n + b^n = (a+b)(a^{n-1} - ba^{n-2} + \cdots + b^{n-1}$ for odd n only, and we have $a^n - b^n = (a-b)(a^{n-1} + ba^{n-2} + \cdots + b^{n-1})$ for all n. So, our above expression simplifies further to:

   $$= \frac{10^{2m+1} - 1}{10 - 1} \cdot \frac{10^{2m+1} + 1}{10 + 1} = \frac{(10-1)(10^{2m} + 10^{2m-1} + \cdots + 1)}{10 - 1} \cdot \frac{(10+1)(10^{2m} - 10^{2m-1} + \cdots + 1)}{10 + 1}$$

   $$= (10^{2m} + 10^{2m-1} + \cdots + 1)(10^{2m} - 10^{2m-1} + \cdots + 1)$$

   where both of the given expressions are positive integers greater than 1. Putting it all together, this means that all the values are composite, except for 101. (We can check that 101 is prime because $\sqrt{101} < 11$ and it is not divisible by $2, 3, 5$ or 7.)

2 Basic Modular Arithmetic Solutions

1. Suppose $\overline{x_n \cdots x_1 x_0} \equiv 0 \pmod{13}$. For what values k is it true that $\overline{x_n \cdots x_1} - kx_0 \equiv 0 \pmod{13}$?

   The first order of business is to rewrite this in a way that we can actually operate on it. If we let $\overline{x_n \cdots x_1 x_0} = X$, then $\overline{x_n \cdots x_1} = (X - x_0)/10$. Now, we have to fix the /10, since we can't "divide" in modular arithmetic. So, we find $10^{-1} \pmod{13}$, which we can easily check is 4, as $4 \cdot 10 = 40 = 39 + 1 = 3 \cdot 13 + 1$. Then, we just have to rewrite and solve:

   $$X \equiv 0 \pmod{13}, \quad 4(X - x_0) - kx_0 \equiv 0 \implies 4X - 4x_0 - kx_0 \equiv 0$$

   $$\implies 4x_0 + kx_0 \equiv 0 \implies 4 + k \equiv 0 \implies k \equiv -4 \equiv 9 \pmod{13} \implies k = 13n + 9, \forall n \in \mathbb{Z}$$

2. Find the remainder when $1 + 2 + \cdots + 2022$ is divided by 1000.

   As one should know, the sum of the first n consecutive integers is $\frac{n(n+1)}{2}$. So, this sum is $\frac{2022(2023)}{2} = 1011(2023) = 2045254$. The remainder when we divide by 1000 is simply the last 3 digits, which are 254.

3. Let S be a subset of $\{1, 2, 3, \dots, 50\}$ such that no pair of distinct elements in S has a sum divisible by 7. What is the maximum number of elements in S?

   Notice that there are 8 elements congruent to $1 \mod 7$:$\{1, 8, 15, 22, 29, 36, 43, 50$. However, since we cut off at 50, there are only 7 elements congruent to each of $0, 2, 3, 4, 5, 6 \mod 7$. Note that the problem constraint tells us that if we have any one element congruent to $a \pmod 7$, then we must not have any other element congruent to $7 - a \pmod 7$. In order to maximize, we certainly want all 8 elements that are congruent to 1. Then, we can take all the elements congruent to 2, and all the elements congruent to 3, but we must not take any congruent to $4, 5$, or 6. We are left with the elements congruent to 7, of which we may take exactly 1.

   This gives us a total of $8 + 7 + 7 + 1 = 23$ elements in our subset.

4. In year N, the 300^{th} day of the year is a Tuesday. In year $N + 1$, the 200^{th} day is also a Tuesday. On what day of the week did the 100^{th} day of year $N - 1$ occur?

   Remember that a year has 365 days, and a leap year has 366 days. So, there are 65 or 66 more days after the 300^{th} day of year N until the end of the year, and 200 more days until the 200^{th} day of year $N + 1$. This gives a total of 265 or 266 days between day 300 of year N and day 200 of year $N + 1$. Since both days are Tuesdays, the number of days separating them must be divisible by 7. We can easily check that this number must then be 266, which means that year N is a leap year.

   Most importantly, this means that year $N - 1$ must not be a leap year, so there are 265 days from day 100 of year $N - 1$, to the end of year $N - 1$. This means that there are a total of $565 = 265 + 300$ days between day 100 of year $N - 1$ and day 300 of year N. We can reduce this number modulo 7, since weeks repeat every 7 days and we are only concerned with what day of the week day 100 occurs on. It is not hard to check that $565 \equiv 5 \pmod 7$, so day 100 of year $N - 1$ occurs 5 days of the week before day 300 of year N, which is a Tuesday.

   This means that day 100 of year $N - 1$ occurs on a Thursday.

5. Determine the smallest positive integer m such that $m^2 + 7m + 89$ is a multiple of 77.

   The most important thing to realize here is that 77 is not prime. We can simplify:

   $$m^2 + 7m + 89 \equiv m^2 + 7m + 12 \equiv (m + 3)(m + 4) \equiv 0 \pmod{77}.$$

   Then, since $77 = 7 \cdot 11$, we want $m + 3$ divisible by 7 and $m + 4$ divisible by 11, or vice versa. (The total product must be divisible by 77, so if we require either $m+3$ or $m+4$ on its own to be divisible by 77, the best we could do is $m = 73$.) In particular, we are looking for a multiple of 7 and a multiple of 11 which differ by 1. The smallest such numbers are $7 \cdot 3 = 21$ and $11 \cdot 2 = 22$, which means $m + 3 = 21, m + 4 = 22 \implies m = 18$. So our smallest m is 18.

3 Modular Arithmetic Applications and Tools, Solutions

1. For what values of $0 < n \leq 25$ is $\frac{(n-1)!}{n}$ an integer?

   First, note by Wilson's Theorem that if n is prime, then this number will never be an integer. If the number is composite, then we may write it as a product $n = ab$, where $1 < a \leq b < n$. From here, there are two cases:

   (a) If there exists a, b such that $a < b$, then both a and b will appear separately in $(n-1)!$, so n will divide out perfectly and the value will be an integer.

   (b) If no such a, b exist, then we must have $a = b$, and both a and b must be prime (otherwise we could rearrange their prime factors to create $a < b$). So, in this case, n is the square of a prime p. If that prime is > 2, then both p and $2p$ will appear in $(n-1)!$, so n will divide out and the result will be an integer. If the prime is 2, then $n = 4$ and we can check that $3!/4 = 6/4 = 3/2$ is not an integer.

   So, the values for which this value is an integer are: all composite numbers, except 4.

2. What is the value of $3^{2022} \pmod{223}$?

   By Fermat's Little Theorem, we can simplify the exponent $\mod 222$, since 223 is prime. This reduces our problem to 3^{24}. First, find $3^3 = 27$. We finish solving by repeated squaring.

$$3^6 = 27^2 = 400 + 49 + 2 \cdot 7 \cdot 20 = 449 + 280 = 729 \equiv 60 \pmod{223}$$

$$3^{12} \equiv 60^2 = 3600 \equiv 32 \pmod{223}$$

$$3^{24} \equiv 32^2 = (2^5)^2 = 2^{10} = 1024 \equiv 132 \pmod{223}.$$

3. Let $a_n = 6^n + 8^n$. Determine the remainder upon dividing a_{83} by 49.

   Note that by Euler's Totient Theorem, we have $\phi(49) = 42$, so $a^{42} \equiv 1$ whenever $\gcd(a, 49) = 1$. In particular, both 6 and 8 are relatively prime to 49, so $1 \equiv 6^{42} \equiv 6^{84} \equiv 8^{42} \equiv 8^{64} \pmod{49}$. This means that $6^{83} \equiv 6^{-1}$ and $8^{83} \equiv 8^{-1}$. So it just remains to find these inverses. Note that $6 \cdot 8 = 48 \equiv -1 \pmod{49}$, so $6(-8) \equiv 8(-6) \equiv 1 \pmod{49}$. So, $6^{-1} \equiv -8 \pmod{49}$ and $8^{-1} \equiv -6 \pmod{49}$. This gives us $a_{83} \equiv -8 - 6 \equiv -14 \equiv 49 - 14 \equiv 35 \pmod{49}$. So, the remainder will be 35.

4. Find the smallest solution to this system of congruences:

$$\begin{aligned} x &\equiv 1 \mod 2 \\ x &\equiv 2 \mod 3 \\ x &\equiv 3 \mod 5 \\ x &\equiv 4 \mod 7 \end{aligned}$$

   We can solve this by combining the equations two at a time, following the Chinese Remainder Theorem. Start with the first 2. If $x \equiv 1 \mod 2$ and $x \equiv 2 \mod 3$, then $x \equiv 5 \mod 6$. These numbers are smaller, so it's not hard to find this via a quick guess-and-check, or brute force.

   Combining this with the third, we want $x \equiv 3 \mod 5$. The possible numbers we have for 5 mod 6 are $5, 11, 17, 23, 29$ of which only $23 \equiv 3 \mod 5$. So, we can replace the first 3 congruences with $x \equiv 23 \mod 30$.

   Finally, we want $x \equiv 4 \mod 7$. Note that $30 \equiv 23 \equiv 2 \mod 7$, so we want k such that $30k + 23 \equiv 2k + 2 \equiv 4 \mod 7$, which gives us $k = 1$, so $x \equiv 30 + 23 \equiv 53 \mod 210$.

   So, the smallest solution x is 53.

4 Using Modular Arithmetic, Solutions

1. Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

6 is small, so we can check the cubes $\mod 6$ easily, and we can confirm that $a^3 \equiv a$ for all a mod 6. So our sum is equal to 2018^{2018}. Note that $2018 \equiv 2 \mod 6$, as it is even, and leaves a remainder of 2 when divided by 3. So, we are really looking for $2^{2018} \pmod 6$. We cannot apply Euler's Totient theorem, as $\gcd(2, 6) = 2 > 1$, but we can see that there is a pattern to powers of 2:

$$2 \equiv 2, \quad 2^2 \equiv 4, \quad 2^3 \equiv 2, \quad 2^4 \equiv 4 \ldots$$

And since 2018 is even, this means that $2018^{2018} \equiv 2^{2018} \equiv 4 \pmod 6$.

2. Let $N = 123456789101112 \ldots 4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

45 is a bit unwieldy to work with as a modulus, but we can split it up into 9 and 5 and work with those via the chinese remainder theorem. Notice that $N \equiv 4 \pmod 5$, and that the sum of the digits, $\mod 9$ will be congruent to $1 + 2 + \ldots + 44 = \frac{44 \cdot 45}{2} = 22 \cdot 5 \cdot 9 \equiv 0 \mod 9$. So, $N \equiv 4 \pmod 5$ and $N \equiv 0 \pmod 9$, and the smallest number satisfying both of these conditions is 9.

That means the remainder when we divide N by 45 is 9, by the Chinese Remainder Theorem.

3. There is a pile of eggs. Joan counted the eggs, but her count was off by 1 in the 1's place. Tom's count was off by 1 in the 10's place. Raoul was off by 1 in the 100's place. Sasha, Jose, Peter, and Morris all counted the eggs and got the correct count. When these seven people added their counts together, the sum was 3162. How many eggs were in the pile?

Let $J, T, R \in \{1, -1\}$, where J is the correction we would have to make to Joan's count, $10T$ is the correction we would make to Tom's count, and $100R$ is the correction we would make to Raoul's count. Since the correct sum is divisible by 7, we have:

$$3162 + J + 10T + 100R \equiv 0 \pmod 7$$

Note that $3162 \equiv 5$, $100 \equiv 2$, and $10 \equiv 3$, $\mod 7$. Our equation now becomes:

$$5 + J + 3T + 2R \equiv 0 \implies J + 2R + 3T \equiv 2 \pmod 7$$

Since each person was only off by 1 in their incorrect digit, we have $J = T = 1$ and $R = -1$. So the correct sum is $3162 + 1 + 10 - 100 = 3073$. To find the total eggs, we divide by 7 to get 439.

4. Find the least positive integer n for which $2^n + 5^n - n$ is a multiple of 1000.

This solution is incredibly tricky, and I left this as a harder challenge problem. However, the main idea here is to (implicitly) use Chinese Remainder theorem, and to also heavily use the fact that $a \equiv b \pmod{m} \implies a \equiv b \pmod{d}$, wherever d divides m.

We have that $2^n + 5^n \equiv n \pmod{1000}$, so $2^n + 5^n \equiv n \pmod{8}$ and $2^n + 5^n \equiv n \pmod{125}$. It is easy to check $n < 3$ don't work, so $n \geq 3$. Then, $2^n \equiv 0 \pmod{8}$ and $5^n \equiv 0 \pmod{125}$, so we just have $5^n \equiv n \pmod{8}$ and $2^n \equiv n \pmod{125}$. Let us consider both of these congruences separately.

First, focus on $5^n \equiv n \pmod{8}$. By Fermat's Little Theorem, we have $5^4 \equiv 1 \pmod{8}$, so $5^5 \equiv 5 \pmod{8}$. Note that this pattern holds, where $5^n \equiv 1 \pmod{8}$ for even n, and $5^n \equiv 5 \pmod{8}$ for odd n. So, the only solutions to this equation occur when $n \equiv 5 \pmod{8}$, or $n = 8k + 5$.

Now, we look at $2^n \equiv n \pmod{125}$. By what we stated above, this requires $2^n \equiv n \pmod{25}$ and $2^n \equiv n \pmod{5}$. Plugging in $n = 8k + 5$, we get $2^{8k+5} \equiv 8k + 5 \pmod{5} \implies 2^{8k} \cdot 32 \equiv 8k \pmod{5}$. By Fermat's Little Theorem, $2^4 \equiv 1 \implies 2^8 \equiv 1 \implies 2^{8k} \equiv 1 \pmod{5}$. So we have $32 \equiv 2 \equiv 3k \pmod{5} \implies k = 5m + 4$.

Then, $n = 8(5m+4)+5 = 40m+37$. Plugging in again, we get $2^{40m+37} \equiv 40m-3 \pmod{125} \implies 2^{40m+37} \equiv 40m - 3 \pmod{25}$. Euler's Totient Theorem tells us that $2^{20} \equiv 2^{40} \equiv 1 \pmod{25}$, so we get $2^{37} \equiv 2^{-3} \equiv 15m + 12 \pmod{25}$. Multiplying both sides by $2^3 = 8$, we have $1 \equiv 120m + 96 \equiv 20m + 21 \implies 20m \equiv 4 \pmod{25}$. This happens precisely when $m \equiv 4 \pmod{5}$, aka $m = 5s + 4$. Plugging in for m, we have $m = 200s + 197$.

Now, we are finally ready to plug n into the congruence modulo 125. As before, Euler's totient theorem tells us that $2^{100} \equiv 1 \pmod{25}$. So, we have $2^{200s+197} \equiv 2^{197} \equiv 2^{-3} \equiv 200s + 197 \equiv 75s + 72$. Multiplying both sides by 8 gives us $1 \equiv 600s + 576 \equiv 100s + 76 \implies -75 \equiv 50 \equiv 100s \pmod{125}$, which happens precisely when $s \equiv 3 \pmod{5}$, or $s = 5x + 3$.

Finally, plugging this in, we find that $n = 200(5x + 3) + 197 = 1000x + 600 + 197 = 1000x + 797$. So, the smallest positive value of n is 797.