ORMC AMC Group: Week 2
AMC Basics

October 2, 2022

1 Algebraic Manipulations Solutions

1. Given real numbers a, b with $a + b = 9$ and $ab = 7$, find $a^3 + b^3$.

   Using the sum of cubes factoring identity, our desired answer is $(a+b)(a^2 - ab + b^2)$. We already have $a + b$, and $(a + b)^2 = a^2 + 2ab + b^2$, which is $3ab + (a^2 - ab + b^2)$. Putting this all together, we have:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) = 9((a + b)^2 - 3ab) = 9(9^2 - 3(7)) = 9(81 - 21) = 540.$$

2. Given $3x + \frac{1}{2x} = 3$, find $8x^3 + \frac{1}{27x^3}$.

   The trick here is to realize that since terms both involve x, we can multiply by $\frac{2}{3}$ to "swap" them. That is, $2x + \frac{1}{3x} = \frac{2}{3}(3x + \frac{1}{2x}) = \frac{2}{3}(3) = 2$. Then, using our work from the previous exercise, we can rewrite our desired answer as:

$$(2x)^3 + \left(\frac{1}{3x}\right)^3 = \left(2x + \frac{1}{3x}\right)\left(\left(2x + \frac{1}{3x}\right)^2 - 3(2x)\left(\frac{1}{3x}\right)\right) = (2)\left(2^2 - 3 \cdot \frac{2}{3}\right) = 4.$$

3. Find all pairs of positive integers (a, b) with $a \leq b$ such that $\frac{1}{a} + 1b = 16$

   Multiply through by $6ab$ rearrange to get a slightly more familiar form: $ab - 6a - 6b = 0$. The point of this is now we can add $6 \cdot 6 = 36$ to get into the factorable form:

$$36 = ab - 6a - 6b + 36 = (a - 6)(b - 6).$$

   From here, it is just a prime factorization problem, since a, b are positive integers. There are 9 factors of 36, so there are 5 factor pairs: $(1, 36), (2, 18), (3, 12), (4, 9), (6, 6)$. Each of these corresponds to exactly one pair (a, b), which we can get by adding 6 to each of the numbers. Our answers are:

$$(7, 42), (8, 24), (9, 18), (10, 15), (12, 12).$$

4. Evaluate $2022^3 - 2022^2 \cdot 2023 - 2022 \cdot 2023^2 + 2023^3$

   The presentation of the expression is a bit misleading, as it seems to suggest more complex factoring at play, but combining some terms shows us that this is just a difference of squares problem:

$$= 2022^2(2022) - 2022^2(2023) - (2023^2(2022) - 2023^2(2023))$$
$$= 2022^2(2022 - 2023) - (2023^2(2022 - 2023))$$
$$= -2022^2 + 2023^2 = 2023^2 - 2022^2 = (2023 + 2022)(2023 - 2022) = 4045.$$

2 Equation Techniques Solutions

1. Find all x such that

$$\frac{1}{\sqrt{x-5}+1}+3=\frac{4}{\sqrt{x-5}+1}$$

Isolate x:

$$\implies 3=\frac{3}{\sqrt{x-5}+1} \implies 1=\frac{1}{\sqrt{x-5}+1} \implies \sqrt{x-5}+1=1 \implies \sqrt{x-5}=0$$

$$\implies x-5=0 \implies x=5.$$

2. Find all pairs (a, b) such that

$$a(a-7b)=60$$
$$b(7b-a)=20$$

We can start by dividing the two equations, since we notice $a-7b=-(7b-a)$. Dividing yields $a/b=-3 \implies a=-3b$. Plugging in to the first equation,

$$-3b(-3b-7b)=3b(10b)=30b^2=60 \implies b=\pm\sqrt{2}$$

Since $a=-3b$, our solutions are $(3\sqrt{2},-\sqrt{2}),(-3\sqrt{2},\sqrt{2})$.

3. Suppose that x, y, z are real numbers such that

$$x=y+z+2$$
$$y=z+x+1$$
$$z=x=y+4$$

First, we rewrite the system as:

$$-2=-x+y+z$$
$$-1=x-y+z$$
$$-4=x+y-z$$

Notice that each variable appears with a positive sign twice, and with a negative sign once. So, if we add all the equations together, we will get $x+y+z$ on the right-hand side. That is:

$$-7=x+y+z.$$

4. Solve the following system of equations:

$$2x_1+x_2+x_3+x_4=1,$$
$$x_1+2x_2+x_3+x_4=2,$$
$$x_1+x_2+2x_3+x_4=3,$$
$$x_1+x_2+x_3+2x_4=4$$

If we can get rid of $x_1+x_2+x_3+x_4$ from each equation, we will get each variable explicitly. Since there is a symmetry to the system of equations, we can add them all together to get some multiple of this expression. That is, if we add them together, we get:

$$5x_1+5x_2+5x_3+5x_4=10 \implies x_1+x_2+x_3+x_4=2$$

Subtracting this from each of the equations, we get

$$x_1=1-2=-1,$$
$$x_2=2-2=0,$$
$$x_3=3-2=1,$$
$$x_3=4-2=2.$$

3 General Function Solutions

1. A function is defined recursively by

$$f(1) = f(2) = 1,$$
$$f(n) = f(n-1) - f(n-2) + n, \; n \geq 3.$$

Find $f(2018)$.

As a general tip, it helps to try writing out terms like $f(n-1), f(n+1)$ for recursive functions to see if something cancels out. For instance, this function gives us:

$$f(n+1) = f(n) - f(n-1) + n + 1.$$

If we add this to the equation given in the problem, we get

$$f(n+1) = -f(n-2) + 2n + 1.$$

Notice that we had f three times in the original equations, and now we have it only twice. The main issue is the n, so we can try to get rid of that by writing out our new expression for $f(n+4)$ (plug in n=n+3):

$$f(n+4) = -f(n+1) + 2n + 6 + 1.$$

Subtracting the two equations, we get:

$$f(n+4) - f(n+1) = f(n-2) - f(n+1) + 2n + 6 + 1 - 2n - 1 \implies f(n+4) = f(n-2) + 6.$$

Re-indexing gives us $f(n+6) = f(n) + 6$. Then since $2018 = 6 \cdot 336 + 2$, we can write

$$f(2018) = f(2 + 6 \cdot 336) = f(2) + 6 \cdot 336 = 1 + 2016 = 2017.$$

2. If $\sum_{n=1}^{2020} f(n) = 0$, and $f(n+1) = \dfrac{f(n)-1}{f(n)+1}$, find $f(1)$.

Plugging in a few times, we find that $f(2) = \frac{f(0)-1}{f(0)+1}, f(3) = -\frac{1}{f(0)}, f(4) = -\frac{f(0)+1}{f(0)-1}, f(5) = f(0)$. In particular, this function repeats every 4 integers, so we can rewrite the sum as

$$505 \sum_{n=1}^{4} f(n) = 0 \implies \sum_{n=1}^{4} f(n) = 0$$

In particular, we only have to sum up the first 4 terms. For convenience, let $x = f(1)$. Summing up the first four terms (equal to 0), we have

$$x + \frac{x-1}{x+1} - \frac{1}{x} - \frac{x+1}{x-1} = 0 \implies x + \frac{x-1}{x+1} = \frac{1}{x} + \frac{x+1}{x-1} \implies \frac{x^2+x+x-1}{x-1} = \frac{x-1+x^2+x}{x(x-1)}$$

Note that the numerators are both equal to $x^2 + 2x - 1$. This gives us two possibilities: either both numerators are equal to 0, or the denominators are equal.

If the numerators are equal to 0, then x is a solution to the quadratic equation $x^2 + 2x - 1$, so we can use the quadratic formula to get $x = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$.

If the numerators are nonzero, then the denominators are equal, so $x + 1 = x(x-1) \implies x^2 - 2x - 1 = 0$. So, using the quadratic formula, we get $x = \frac{2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$.

Thus, our solutions are $x \in \{1+\sqrt{2}, 1-\sqrt{2}, -1+\sqrt{2}, -1-\sqrt{2}\}$

3. If $f(x) = \dfrac{x}{x+1}$, what is $f(f(f(f(2009))))$?

We can use a different recursive function strategy here, which is testing some values to try to find a pattern. Note that

$$f(f(x)) = \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} = \frac{\frac{x}{x+1}}{\frac{x+x+1}{x+1}} = \frac{x}{2x+1}$$

Notice that the coefficient of x in the denominator increased by 1. We can hypothesize that this will happen every time. To check our hypothesis, think about $f(\frac{x}{nx+1})$. Plugging in:

$$f(\frac{x}{nx+1}) = \frac{\frac{x}{nx+1}}{\frac{x}{nx+1}+1} = \frac{\frac{x}{nx+1}}{\frac{x+nx+1}{nx+1}} = \frac{x}{(n+1)x+1}$$

So, $f(f(f(x)) = f(\frac{x}{2x+1}) = \frac{x}{3x+1}$, and $f(f(f(f(x)))) = f(\frac{x}{3x+1}) = \frac{x}{4x+1}$. So our answer is

$$f(f(f(f(2009)))) = \frac{2009}{4 \cdot 2009 + 1} = \frac{2009}{8037}$$

4 Logarithm Solutions

1. Log properties proofs:

   - $\log_b(x) - \log_b(y) = \log_b(x/y)$
     - Proof: $b^{\log_b(x)-\log_b(y)} = b^{\log_b(x)}/b^{\log_b(y)} = x/y = b^{\log_b(x/y)} \implies \log_b(x) - \log_b(y) = \log_b(x/y)$.
   - $\log_b(x^y) = y\log_b(x)$
     - Proof: $b^{y\cdot\log_b(x)} = (b^{\log_b(x)})^y = x^y = b^{\log_b(x^y)} \implies \log_b(x^y) = y\log_b(x)$.
   - $\log_b(1) = 0$
     - $b^{\log_b(1)} = 1 = b^0 \implies \log_b(1) = 0$

2. What is the value of $\sqrt{\log_2(6) + \log_3(6)}$

   Before I get right into this, a little aside. We often go over lots of general-case formulas, and most math classes will go over lots of general-case formulas. This is because they always work, and in most cases, they will be most useful. However, that is not to say that special cases are not useful. Often, it can be incredibly useful to be familiar with special cases, since the general case will not "show up" in an obvious way every time, and it is helpful to be able to recognize special cases.

   Of course, it's not practical for any class to go over every special case, and it probably isn't convenient for you to try to list out every possible special case, and sit around wondering if you missed any. A lot of the time, you can get a good handle on special cases just by playing around with the general-case equations, and as you do more practice problems, you will get better at recognizing both the special cases and the general case as they show up.

   The particular special case we are interested in here is a special case of squaring a binomial:

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2(x)(1/x) + \frac{1}{x^2} = x^2 + 2 + \frac{1}{x^2}$$

   Notice how the middle term cancels out and is simply 2. This is something to look out for: if you have $2 + a + \frac{1}{a}$, it is a perfect square, and its square root is $\sqrt{a} + \sqrt{\frac{1}{a}}$.

   Applying log rules to our original expression, we get:

$$\sqrt{\log_2(6) + \log_3(6)} = \sqrt{\frac{\log(6)}{\log(2)} + \frac{\log(6)}{\log(3)}} = \sqrt{\frac{\log(3)}{\log(2)} + \frac{\log(2)}{\log(2)} + \frac{\log(3)}{\log(3)} + \frac{\log(2)}{\log(3)}}$$

$$= \sqrt{\frac{\log(3)}{\log(2)} + 2 + \frac{\log(2)}{\log(3)}}$$

   Since $\log(3)/\log(2)$ is the reciprocal of $\log(2)/\log(3)$, we can now apply our special case:

$$= \sqrt{\left(\sqrt{\frac{\log(3)}{\log(2)}} + \sqrt{\frac{\log(2)}{\log(3)}}\right)^2} = \sqrt{\frac{\log(3)}{\log(2)}} + \sqrt{\frac{\log(2)}{\log(3)}} = \sqrt{\log_2(3)} + \sqrt{\log_3(2)}$$

   So the correct answer is (d).