AMC 8 Training - Word Problems and Number Theory

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1 Introduction

Why are there words in my math problem?

Word problems generally test your ability to understand what words mean in the context of math. They can be answered by translating the English words presented into math.

Well that's great and all...but what is Number Theory?

Number Theory is the study of how numbers, specifically integers, work. This includes prime factorization, divisibility, and patterns in digits and decimals.

So now that we understand that...let's take a look at an example word problem.

2 Word Problems

Here is the problem we will work through together.
- - - -
John has 5 times more apples than Jane. If Jane has 4 apples, then how many apples does John have?
- - - -
In this case, we can see that the question is asking about a quantity of apples. In math, we represent quantities using variables. So, let's do just that. We can say that John has x apples and Jane has y apples.

Now, to calculate x, we look at what we know. John has 5 times more apples than Jane, meaning that we can write that $x = 5y$. We know that Jane has 4 apples, so $y = 4$, meaning that

$x = (5)(4) = 20$

We now know that John has 20 apples.

Therefore, we can rewrite our word problem into the equation: If $x = 5y$, and $y = 4$, then $x =$?
- - - -

This was a very simple example, but the idea of turning math problems into math equations is the best way to go about solving word problems.

2.1 Exercise 1

Abby, Bob, and Cathy each have some marbles. Abby has 36 and Bob has 22. Abby gives half her marbles to Bob, who gives half his combined marbles to Cathy. In the end, Cathy has 29 marbles. How many marbles did Cathy start with?

3 Prime Numbers

Prime Numbers are numbers that **don't have** any other factors other than one and itself. Some examples are 2, 3, and 89.
Composite Numbers are numbers that **have** other factors other than one and itself. Examples include 4, 15, and 120.

3.1 Exercise 1

Using your knowledge of divisibility from the first week, find all the prime numbers in the following grid. Use the following instructions:

1. Cross out 1 by **Shading** in the box completely. One is neither prime nor composite.
2. Use a forward **Slash** to cross out all multiples of 2 (starting with 4 since 2 is prime).
3. Use a backward **Slash** / to cross out all multiples of 3 (starting with 6 since 3 is prime).
4. Multiples of 4 have been crossed out already (multiples of 2).
5. Draw a **Square** on all multiples of 5 (starting with 10 since 5 is prime).
6. Multiples of 6 have been crossed out already (multiples of 2 and 3).

7. **Circle** all multiples of 7 (starting with 14 since 7 is prime).
8. Multiples of 8 have been crossed out already (multiples of 2).
9. Multiples of 9 have been crossed out already (multiples of 3).
10. Multiples of 10 have been crossed out already (multiples of 2 and 5).

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

What numbers did you have left? Write them here.

You just found all the prime numbers less than 100! The method we used was called the *Sieve of Eratosthenes*. It's an ancient algorithm that is cool to know and use, but don't worry: you won't need to remember the name of it.

4 Factors

A *factor* of an integer n can be defined as an integer that can be multiplied by another integer to get this number n.

4.1 Divisibility Rules

A number is divisible by:
2: If it is even.
3: If the sum of its digits is divisible by 3.
4: If the number formed by the last two digits is divisible by 4.
5: If the one's digit is 5 or 0.
6: If it is divisible by 2 and 3 (all even multiples of 3).
7: There is no *good* trick for 7, just do the division.
8: If the number formed by the last three digits is divisible by 8.
9 If the sum of the digits is divisible by 9.
10: If the last digit is a 0.
11: We learned this trick last week, can you write it below?

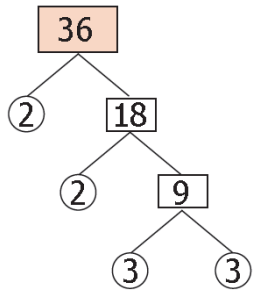
4.1.1 Divisibility Exercises

1. Determine whether the following numbers are divisible by 6 **using divisibility rules**.
   (a) 264
   (b) 975
   (c) 12,560
2. Determine whether the following numbers are divisible by 72 (*hint*: 72 is $8 * 9$) **using divisibility rules**.
   (a) 924
   (b) 1,284
   (c) 14,790

4.2 The Fundamental Theorem of Arithmetic

The Fundamental Theorem of Arithmetic states that every positive integer has a unique prime factorization. For example, $5544 = (2)^3(3)^2(7)(11)$. There is no other way to factor 5544 into a product of primes.

The easiest way to find the prime factorization of a number is to create a factor tree. Here is an example:



Try it yourself! Use a factor tree to find all the factors of **a)** 84 and **b)** 520.

4.3 What can you do with factors?

There will be many different questions the AMC test might ask where factors will be needed, but here are some of the most common usages.

4.3.1 Sum of Factors

The sum of the factors of an integer n, where p_1, p_2, etc. represent the prime factors of n and $(p_1)^a(p_2)^b(p_3)^c...(p_x)^y$ is the prime factorization of n, is

$$(a+1)(b+1)(c+1)...(y+1)$$

Try finding the sum of the factors of the following numbers: **a)** 375 **b)** 360.

4.3.2 Product of Factors

The product of the factors of an integer n, where S is the sum of its factors as defined above, is

$$n^{\frac{S}{2}}$$

Try finding the product of the factors of the following numbers: **a)** 375 **b)** 360.

5 Practice

Lets practice doing word problems and number theory with the following past AMC 8 test problems!