Intermediate 2 Week 3 ORMC: Geometry, Geometric Optimization, and More

Kason Ancelin

October 2022

1 Introduction

0. (Warm Up) Which is larger, 5^{44} or 4^{53}?

Today, we will be studying Geometry and Geometric Optimization. In mathematics, optimization is all about finding extrema, which we can think of as either maximum or minimum values. For instance, real world problems may require you to maximize the grazing area for cows with a fixed amount of fence, maximize profit of a company with certain constraints on production and sale, or minimize the distance traveled along a route.

For many of these geometry problems, sketching the problem and then either making some geometric constructions or deriving an algebraic expression could be useful. Here are some ideas you may want to look back upon while problem solving:

- When deriving an algebraic expression which only works for some x or y values (what are called constraints on the variables), think about graphing the function, and how you would find minimum or maximum values on a interval or across a set of points.
- The restrictions the problem places on its variables (for instance, maybe x and y are non-negative and has other constraints) which create an enclosed shape on which all feasible solutions (points which at least satisfy the constraints) must lie. Then from here, maybe we can seek out points in this enclosed shape to find either a maximum or minimum value.
- Maybe think of a classic geometric manipulation of a point or line for one or two of these problems.

We will solve the following two problems together as an introduction to this topic.
a) Billy is playing a video game in which he starts out with 5 EXP. For every quest he completes, he gains 16 EXP. However, he can only do one quest per day, and every day he automatically loses the total number of quests he has completed in previous days squared EXP. How many quests should Billy play to maximize his EXP.

b) In this problem, we display the idea of a feasible set or feasible solution space on the board, and leave you and your fellow classmates to solve it. Suppose x, y obey all of the following:
- $x, y \leq 0$
- $x + y \leq 10$
- $\frac{1}{2}x + y \leq 10$
- $y \leq 9$
Now, maximize $x + y$.

1. Two 2-digit numbers are written digit-wise as ab and cd. You are given the maximum product of the set $a \times c, a \times d, b \times c, b \times d$ is 9. What is the maximum product of $ab \times cd$ (where ab and cd are again the digit-wise representation of the numbers) and what are the values of a, b, c, d?

2. A rectangle has a perimeter of 30 units. What is the maximal area this rectangle can enclose? More generally, prove what side lengths of a rectangle of fixed perimeter will enclose the maximum area.

3. Find two positive real numbers such that the sum of the first number squared and the second number is 48 and their product is a maximum. How about if the sum of the first number squared and the second number is -3?

4. You and your friend both live on the same side of a river which flows in a perfectly straight line. You are tasked with biking from your house to the river to pick up some cool rocks and then to your friend's house to give him these rocks as a birthday present. Since you are late for a visit to a friend's house, you need to ensure you take the shortest route to get there. Prove what this route is on paper using a geometric construction. (Note: This is one of the oldest optimization problems, Heron's Shortest Distance Problem. It was solved by Heron of Alexandria between 10-75 CE.)

5. Let x denote the number of Reese's made by Hershey's and y the number of Hershey's Bars made by Hershey's. Due to budget cuts, Hershey's can only make at most 10 candy bars this year. Additionally, due to low supply, Hershey's has determined that people will pay \$7 for a Reese's Cup and \$5 for a Hershey's Bar. If Hershey's wishes to maximize their revenue, how many of each candy will they produce?

6. Building on problem 4, now assume that we have the additional constraint that $3x + y \leq 24$ due to shortages of inputs. Now how many of each candy will they produce?

7. We have 12 meters of rope and want to split into two pieces which will become the perimeter of a rectangle and the circumference of a circle. We want to maximize the sum of the area enclosed by the rectangle and circle. How should we split the rope?

8. Suppose you have a square inscribed inside a semicircle. Now you place a square whose bottom side lies on the original inscribed square's top side and this new square has its top two vertices touching the circle. Find the area of this smaller top square.

The following exercises use similar triangles to prove their results. Please do the challenge problems first, as they are more similar to the rest of this worksheet (and probably more fun). We strongly encourage you to finish every other problem on this worksheet before completing problems 9 and 10. Problems 9 and 10 are more low level Olympiad style problems, thus their plain nature and statement of problem.

9. Let ABC be a triangle with $BC = 10$, $AC = 12$. Point E is chosen on the side AC so that $AE = 4$. Point K is chosen on the extension of the ray BE so that AK is parallel to BC. What is AK?

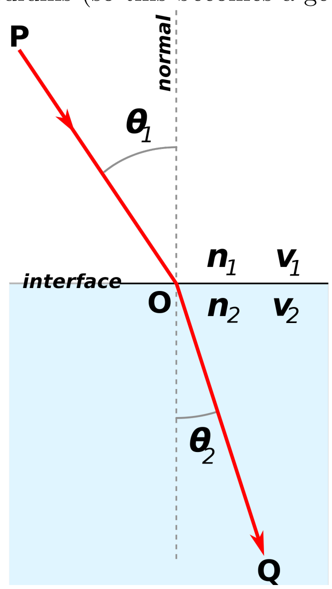
10. The point D is chosen on the side AC of triangle ABC. If $AB = 6$, $AD = 4$, $BAC = 40$, $ACB = 60$, $DBC = 20$, find DC.

2 Challenge Problems

1. You are playing a game in which you roll two pairs of two dice. Your goal is to get the highest ending value possible. You can either take the minimum of the maximum of each pairs' rolls, or the maximum of the minimum of each pairs' rolls. Is one of these strategies better? If so, prove which one. Furthermore, what about three or more pairs of dice? Is one of the strategies better?

2. You have a 5-inch by 8-inch index card. You will cut equal sized squares out from each corner and then fold the paper to create a box with an open upper face. What size square should you cut to maximize the volume of this resulting box? (hint: the graph of a function might help with this one)

3. Prove Snell's Law, given that you know light wants to travel between P and Q as fast as possible but can only move with speed v_1 and v_2 in the different mediums (so this becomes a geometric optimization problem).



(In physics, Snell's law states that, for a given pair of media, the ratio of the sines of angle of incidence (θ_1) and angle of refraction (θ_2) is equal to the refractive index of the second medium with respect to the first (n_{21}) which is equal to the ratio of the refractive indices (n_2/n_1) of the two media, or equivalently, to the ratio of the phase velocities (v_1/v_2) in the two media: $\dfrac{\sin\theta_1}{\sin\theta_2} = n_{21} = \dfrac{n_2}{n_1} = \dfrac{v_1}{v_2}$)

(We will help a good amount with this if needed).