OLGA RADKO MATH CIRCLE: ADVANCED 3

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Worksheet 2: Symmetry Groups I

During the first two weeks we focused on groups. A group, is a set G with a binary operation $*$ and a unit 1_G for the operation, such that $*$ is associative and every element has an inverse. We also studied several other objects that helped us get to the definition of group. In this worksheet, we will study certain groups associated to the symmetries of a geometric object. These are the so-called *symmetry groups*. The concept of symmetry groups will be formally defined below.

Definition 1. Let $X = \{A, B, C, D\}$ be a set with four elements. A permutation of X is a rearrangement of its elements. To write down this rearrangement, we use the following notation with parentheses:

$$\sigma_1 = (AB)(CD).$$

The previous notation means that the permutation σ_1 sends A to B, C to D, B to A, and D to C. You have to think about (AB) as a *cycle* that sends A to B and B to A. Analogously, if we write

$$\sigma_2 = (ABC)(D)$$

it means that the permutation σ_2 sends A to B, B to C, C to A and D to D. Here, (ABC) represents a cycle of three elements. In this case, we often suppress D from the notation, as it is just a cycle of length one: it sends D to D and does nothing else. We can concatenate permutations to obtain

$$\sigma_2\sigma_1 = (ABC)(D)(AB)(CD) = (ACD)(B) = (ACD).$$

The set of permutations of X with the operation "concatenation" is a group, which is called S_4[1]. The group S_4 has 24 elements. The element σ_1 satisfies that $\sigma_1^2 = \sigma_1 * \sigma_1 = 1$, because if we apply σ_1 twice, then the corresponding permutation is just the identity.

Problem 2.0: Consider the group S_4 just defined.

- Find all the permutations $\sigma \in S_4$ for which $\sigma^2 = 1$.
- Find all the permutations $\sigma \in S_4$ for which $\sigma^4 = 1$.

Solution 2.0:

[1] The subindex 4 represents the number of elements in the set X.

Problem 2.1: Consider a square with vertices A, B, C and D as shown in the following picture:

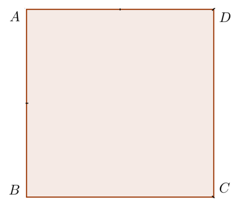


Figure 1: Square.

Find 8 different symmetries (rotations, reflections) of the square.
For instance, you can reflect along the y-axis, x-axis, etc.

- Can you come out with 6 other symmetries of the square?
- Can you find a 9th symmetry that is different from the previous ones?
- Describe the symmetries of the square as permutations of the vertices $\{A, B, C, D\}$.

Solution 2.1:

Definition 2. Let $\mathbb{R}^n$ be the n-dimensional real space, i.e., a space with n real dimensions. For instance, $\mathbb{R}^1$ is just the real line, $\mathbb{R}^2$ is the plane, $\mathbb{R}^3$ is the 3D space, etc. Inside $\mathbb{R}^n$ we can consider a geometric object X (like the square in $\mathbb{R}^2$ from the previous problem). A *symmetry* of X is a rotation or reflection in $\mathbb{R}^n$ that preserves X. This means that we have a transformation $s\colon \mathbb{R}^n \to \mathbb{R}^n$ for which $s(X) = X$. Given two symmetries s_1 and s_2 of X, we write $s_1 * s_2$ for the *concatenation symmetry*, i.e., the symmetry obtained by applying s_2 followed by s_1.

Problem 2.2: Let $X \subset \mathbb{R}^2$. Write $\mathcal{S}(X)$ for the set of symmetries of X. Show that $(\mathcal{S}(X), *)$, the set $\mathcal{S}(X)$ with the concatenation operation is a group. What is the identity in this case?
Solution 2.2:

Problem 2.3: Consider the square in Figure 1. Let r be the counter clock-wise rotation of 90 degrees and let t be the reflection with respect to the y-axis (Assume the center of the square is $(0,0)$). Show that every symmetry of the square can be expressed as concatenations of r and t.

Solution 2.3:

Definition 3. Let $(G,*)$ be a group. A group is said to be *finitely generated* if there exists a finite set of elements $g_1,\ldots,g_s$ such that every element $g \in G$ can be written as a product of the g_i's. For instance, in the previous problem, we showed that the group of symmetries of the square is generated by r and t.

Problem 2.4: Answer the following questions regarding finite generation of groups

- Is the group $(\mathbb{Q},+)$ finitely generated?
- Let C be a circle of radius one. Is the group $(\mathcal{S}(C),*)$ of symmetries of the circle finitely generated?

Solution 2.4:

Problem 2.5: Let r and t be the generators of the group of symmetries of the square that we found above. Find 3 different products of t and r that gives the identity. For instance, $t^2 = 1$ in this group.

- Can you find two other such equalities?
- Try to find more products of t and r that gives the identity.
- Can you write these new products as products of t^2, r^4, and $rtrt^{-1}$?

Solution 2.5:

Definition 4. Let $(G,*)$ be a finitely generated group. Let $g_1,\dots,g_s$ be the generators of G. A *relation* among the g_i's is a product of these objects which equals the identity 1_G. The *set of relations* of $g_1,\dots,g_s$ is the set of all possible relations among them. We say that a set of relations $r_1,\dots,r_k$ *generates all relations* if every relation can be written as products of the r_i's.

For instance, in Problem 2.5 we showed that r^4, t^2, and $rtrt^{-1}$ generates all relations among r and t.

Definition 5. A *presentation* of a group G is a way of writting the group as follows:

$$G := \langle g_1,\dots,g_s \mid r_1,\dots,r_k\rangle,$$

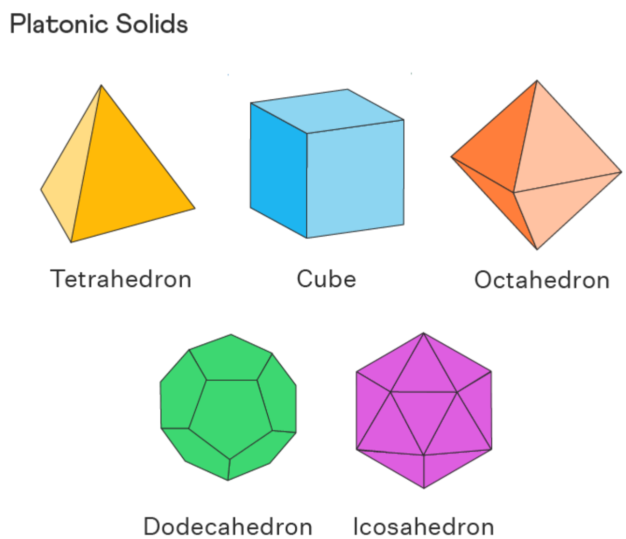
such that $g_1,\dots,g_s$ are generators of G and the $r_i,\dots,r_k$ is a finite set of generating relations.

Problem 2.6: Let $(\mathbb{Z}_n,+)$ be the integers modulo n with addition. Find a presentation for this group. This group is usually called the *cyclic group* with n elements and denoted by A_n.
Solution 2.6:

Problem 2.7: Let P_n be a regular n-gon. Find a presentation of $\mathcal{S}(P_n)$.
Solution 2.7:

Definition 6. The group $\mathcal{S}(P_n)$ of the symmetries of a regular n-gon is said to be a *dihedral group* and denoted by D_n.

Definition 7. A *platonic solid* is a geometric solid in $\mathbb{R}^3$ whose faces are all identical, regular polygons meeting at the same three-dimensional angles. There are 5 platonic solids:



The tetrahedron, cube and octahedron were known to Pythagoras back in 500 b.C. However, the first one to describe all the platonic solids and understand its symmetries was the Athenean phylosopher Plato in 420 b.C.

Problem 2.8: Lets denote by T the tetrahedron.

- Show that any symmetry of T induces a permutation of its 4 vertices.
- Show that any permutation of its 4 vertices T can be obtained by a symmetry of T.
- Deduce that the group $\mathcal{S}(T)$ is equivalent[2] to the group of permuations of the 4 vertices.
- Find a presentation for $\mathcal{S}(T)$. Try to generate it with plane reflections that fix two points and permute the other two points.

Solution 2.8:

[2]Equivalent means that for each symmetry of T we can find a unique permutation that corresponds to it.

Problem 2.9: Describe the symmetry group of the octahedron.
Solution 2.9:

Problem 2.10: Describe the symmetry group of the icosahedron.
Solution 2.10:

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