Special Relativity

Compiled by Nikita with problems from Aaron and Koffi

1 More time contraction

Recall that for any observer all the measurements of time change when observing an object with speed v by the factor (also called a Lorentz factor) $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Problem 1. A certain species of bacteria doubles its population every 20 days. Two bacteria are placed in a spacecraft and sent into space. The speed of the spacecraft is $v = 0.995c$.

1. What is the time measured on Earth that corresponds to a doubling of the population?
2. How many times does the population double after 1,000 Earth days?
3. What is the number of bacteria on board the spacecraft after 1,000 days measured in relation to the Earth?

Solution.

1. 20 days corresponds to the proper time in the spacecraft reference frame. The time on Earth T is therefore:$T = \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{20}{\sqrt{1-0.995^2}} = 200 days.$
2. In $1000 = 5 \times 200$ Earth days, there were 5 doublings in the spacecraft.
3. Since at the start, there are 2 bacteria, after 1,000 Earth days, i.e. 5 doublings, there are: $2^{1+5} = 64$ bacteria on board the spacecraft.

2 Length Contraction

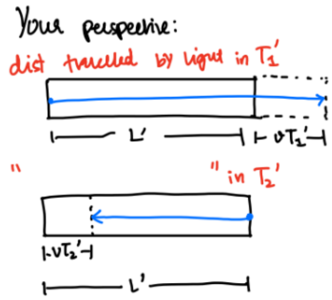
Having seen that strange things happen to the perception of time, let's see what happens to space.

Problem 2. Let's say that Einstein rotates the light clock, so that now it is pointing horizontally along the x-axis - the direction of motion of the train. This will end up affecting your measurement of the length of the light clock, but by how much?

Say that Einstein measures the length of the light clock to be L, and you measure it to be L'. Then suppose he starts the light clock at $t = t' = 0$, while the source of the light is at $x = x' = 0$, you see the light hit the other end at time $t' = T'_1$, and take T'_2 more time to return to the detector (so it hits it at time $t' = T'_1 + T'_2$).

1. Draw a picture of this setup from each perspective.

2. What is the x'-position of the light when it hits the mirror at time T_1'? Solve for T_1' in terms of L' and v.
3. What is the x'-position of the light when it returns to the detector at time $T_1' + T_2'$?
4. At what time does Einstein see the light return to the detector?
5. Using our time dilation formula, at what time do you see the light return to the detector?
6. Solve for the total distance the light travelled (by your measurement), and thus L'/L.



$$\therefore cT_1' = L' + vT_1' \Rightarrow T_1' = \frac{L'}{c-v}$$

$$\therefore cT_2' = L' - vT_2' \quad \therefore T_2' = \frac{L'}{c+v}$$

$$\therefore \text{total time (you)} = \frac{L'}{c-v} + \frac{L'}{c+v} = L'\left(\frac{2c}{c^2-v^2}\right)$$

$$\text{total time (einstein)} = 2L/c$$

$$\text{we know: } \frac{\text{time taken (you)}}{\text{time taken (einstein)}} = \gamma$$

$$\therefore \frac{L'\left(\frac{2c}{c^2-v^2}\right)}{2L/c} = \frac{L'}{L}\left(\frac{c^2}{c^2-v^2}\right) = \gamma$$

$$\therefore \frac{L'}{L}\left(\frac{1}{1-v^2/c^2}\right) = \gamma$$

$$\therefore \frac{L'}{L} \cdot \gamma^2 = \gamma$$

$$\therefore \frac{L'}{L} = \frac{1}{\gamma} \quad \square$$

Solution.

- You will measure $x' = L' + vT_1'$, as the light will have moved across the clock of length L', plus the distance the train has moved, which is vT_1'. We find that $L' + vT_1' = cT_1'$, so $T_1' = \frac{L'}{c-v}$.
- It'll have moved to the point $v(T_1' + T_2')$ in time T_2'. Thus $L' + vT_1' = cT_2'$, and $T_2' = \frac{L'}{c+v}$.
- He just sees it traverse a distance of $2L$, so $T = \frac{2L}{c}$.

- You'll measure $T'_1 + T'_2 = \gamma T = \frac{\gamma 2L}{c}$.
- The light will have moved a total distance of $c(T'_1 + T'_2) = L'\left(\frac{1}{c-v} + \frac{1}{c+v}\right) = \frac{2cL'}{c^2-v^2}$, and $T'_1 + T'_2 = \frac{\gamma 2L}{c} = \frac{2L}{\sqrt{c^2-v^2}}$. Thus $\frac{L'}{L} = \frac{\sqrt{c^2-v^2}}{c} = \frac{1}{\gamma}$.

We call the effect observed in this experiment *length contraction*. It is worth noting that nothing changes about distance in the y- or z-directions – the change only happens in the direction of motion.

Problem 3. We've seen what happens when you measure objects that Einstein keeps on the train. If instead you're standing on the platform with a meterstick along the x-axis, how long does Einstein see it as?

Solution. Einstein sees you as moving at velocity $-v$, because you're moving in the opposite direction. He'll observe the same length contraction, with $\gamma = \frac{c}{\sqrt{c^2-(-v)^2}} = \frac{c}{\sqrt{c^2-v^2}}$, the same factor as you see. Thus he sees a meterstick on the platform as having length $\frac{1}{\gamma}$ meters.

3 Hyperbolic functions

Hyperbolic functions are a family of elementary functions which can be expressed via the exponential function and closely resemble trigonometric functions. We define

$$\cosh\phi = \frac{e^x + e^{-x}}{2} \quad \text{(hyperbolic cosine)}, \qquad \sinh\phi = \frac{e^x - e^{-x}}{2} \quad \text{(hyperbolic sine)},$$
$$\tanh\phi = \frac{\sinh\phi}{\cosh\phi} \quad \text{(hyperbolic tangent)}, \qquad \coth\phi = \frac{\cosh\phi}{\sinh\phi} \quad \text{(hyperbolic cotangent)}.$$

Problem 4. Plot the above hyperbolic functions.

Problem 5 (Pythagorean hyperbolic identity)**.** Prove that $\cosh^2\phi - \sinh^2\phi = 1$.

Problem 6. Express $\sinh(x \pm y)$ and $\cosh(x \pm y)$ via $\sinh x$, $\cosh x$, $\sinh y$ and $\cosh y$.

Problem 7. Express $\tanh(x \pm y)$ via $\tanh x$ and $\tanh y$.

Problem 8. Inverse hyperbolic functions are denoted by Arsinh, Arcosh, Artanh and Arctanh. Express $\operatorname{Arsinh} x$, $\operatorname{Arcosh} x$ and $\operatorname{Artanh} x$ using the natural logarithm function $\ln(x)$ and x.

4 Lorentz Transformations

We can now find a relativistic analog for the equation

$$(t', x', y', z') = (t, x + vt, y, z)$$

found in Galilean relativity.

Throughout, we will assume that at time $t = t' = 0$, you and Einstein agree on the location of the origin $(x, y, z) = (x', y', z')$, and that at all times, $y' = y$ and $z' = z$. Thus we'll just focus on calculating (t', x') in terms of (t, x) and vice versa.

Now say that an event happens along the x-axis, and Einstein measures its coordinates as (t, x), while you measure its coordinates as (t', x').

Problem 9. Based on what you observe at time t', find a formula for x' in terms of x and t'. Solve for x in terms of (t', x').

> **Solution.** At time $t' = 0$, you will see Einstein's origin at vt', and the event at distance $\frac{x}{\gamma}$ away from that, by length contraction. Thus $x' = vt' + \frac{x}{\gamma}$, and $x = \gamma(x' - vt')$.

Problem 10. When Einstein observes the event at time t, how far away is the event from the origin in your coordinates, according to his measurements?
Find a formula for x' in terms of t and x.

> **Solution.** This will be the same as the last problem, but with v replaced with $-v$, as Einstein sees you as moving at velocity $-v$, so we'll get $x' = \gamma(x + vt)$.
>
> Einstein will see the distance between your origin and his as vt, and the distance between his origin and the event as x, so he'll measure the whole distance as $x + vt$ - by length contraction, he should assume the distance is $\gamma(x + vt)$ in your frame of reference.

Problem 11. Solve for (t', x') in terms of (t, x).

> **Solution.** We know that $x' = \gamma(x + vt)$, and $x' = vt' + \frac{x}{\gamma}$, so
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> $$\begin{aligned} t' &= \frac{1}{v}\left(\gamma(x + vt) - \frac{x}{\gamma}\right) \\ &= \gamma t + \frac{1}{v}(\gamma - \frac{1}{\gamma})x \\ &= \gamma t + \frac{\gamma^2 - 1}{\gamma v}x \\ &= \gamma(t + \frac{v}{c^2}x) \end{aligned}$$
>
> Thus we get
>
> $$(t', x') = \gamma\left(t + \frac{v}{c^2}x, x + vt\right).$$

This transformation, turning (t, x) into (t', x') or vice versa, is called a *Lorentz transformation*, after Hendrik Lorentz, who discovered a lot of this before Einstein.

Problem 12. Solve for (t, x) in terms of (t', x'). Show that this Lorentz transformation is the same as the previous one, but swapping the sign on v.

> **Solution.** We will get
>
> $$(t, x) = \gamma\left(t - \frac{v}{c^2}x, x - vt\right).$$

Problem 13. Show that if v is much, much smaller than c, then $(t', x') \approx (t, x + vt)$, showing that for very low speeds v, Einsteinian relativity is basically the same as Galilean relativity.

Solution. First, we observe that for low v, $\gamma \approx 1$. Thus $(t', x') = \gamma\left(t + \frac{v}{c^2}x, x + vt\right) \approx \left(t + \frac{v}{c^2}x, x + vt\right)$. The only remaining term of difference is $\frac{v}{c^2}x$, which will be miniscule if v is much smaller than c.

Problem 14. Show that this transformation preserves the quantity $-(ct)^2 + x^2 + y^2 + z^2$.

Solution. The quantities y and z don't change, so we'll just show $x'^2 - (ct')^2 = x^2 - (ct)^2$.

$$\begin{aligned} x'^2 - (ct')^2 &= \gamma^2 \left((x + vt)^2 - (ct + \frac{v}{c}x)^2\right) \\ &= \gamma^2 \left((1 - (v/c)^2)x^2 + (v - c)^2 t^2\right) \\ &= x^2 - c^2 t^2 \end{aligned}$$

Problem 15 (Optional: uses matrices)**.** Write the transformation sending (t, x, y, z) to (t', x', y', z') as a 4×4 matrix.

Problem 16. Suppose Einstein sees another train go by on a parallel track at velocity u. What velocity do you measure this train at?

Solution. Use composition of Lorentz transformations. Written in matrices, if the speed you measure is w, we have

$$\begin{aligned} \gamma_w \begin{bmatrix} 1 & w/c^2 \\ w & 1 \end{bmatrix} &= \gamma_u \begin{bmatrix} 1 & u/c^2 \\ u & 1 \end{bmatrix} \gamma_v \begin{bmatrix} 1 & v/c^2 \\ v & 1 \end{bmatrix} \\ &= \gamma_u \gamma_v \begin{bmatrix} 1 + uv/c^2 & (u+v)/c^2 \\ u + v & 1 + uv/c^2 \end{bmatrix} \end{aligned}$$

so $\gamma_w = \gamma_u \gamma_v (1 + uv/c^2)$, and $\gamma_w w = \gamma_u \gamma_v (u + v)$, so $w = \frac{u+v}{1+uv/c^2}$.