

# Special Relativity

Compiled by Nikita with problems from Aaron and Koffi

## 1 More time contraction

Recall that for any observer all the measurements of time change when observing an object with speed  $v$  by the factor (also called a Lorentz factor)  $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

**Problem 1.** A certain species of bacteria doubles its population every 20 days. Two bacteria are placed in a spacecraft and sent into space. The speed of the spacecraft is  $v = 0.995c$ .

1. What is the time measured on Earth that corresponds to a doubling of the population?
2. How many times does the population double after 1,000 Earth days?
3. What is the number of bacteria on board the spacecraft after 1,000 days measured in relation to the Earth?

## 2 Length Contraction

Having seen that strange things happen to the perception of time, let's see what happens to space.

**Problem 2.** Let's say that Einstein rotates the light clock, so that now it is pointing horizontally along the  $x$ -axis - the direction of motion of the train. This will end up affecting your measurement of the length of the light clock, but by how much?

Say that Einstein measures the length of the light clock to be  $L$ , and you measure it to be  $L'$ . Then suppose he starts the light clock at  $t = t' = 0$ , while the source of the light is at  $x = x' = 0$ , you see the light hit the other end at time  $t' = T'_1$ , and take  $T'_2$  more time to return to the detector (so it hits it at time  $t' = T'_1 + T'_2$ ).

1. Draw a picture of this setup from each perspective.
2. What is the  $x'$ -position of the light when it hits the mirror at time  $T'_1$ ? Solve for  $T'_1$  in terms of  $L'$  and  $v$ .
3. What is the  $x'$ -position of the light when it returns to the detector at time  $T'_1 + T'_2$ ?
4. At what time does Einstein see the light return to the detector?
5. Using our time dilation formula, at what time do you see the light return to the detector?
6. Solve for the total distance the light travelled (by your measurement), and thus  $L'/L$ .

### 3 Hyperbolic functions

Hyperbolic functions are a family of elementary functions which can be expressed via the exponential function and closely resemble trigonometric functions. We define

$$\begin{aligned} \cosh \phi &= \frac{e^{\phi} + e^{-\phi}}{2} && \text{(hyperbolic cosine),} & \sinh \phi &= \frac{e^{\phi} - e^{-\phi}}{2} && \text{(hyperbolic sine),} \\ \tanh \phi &= \frac{\sinh \phi}{\cosh \phi} && \text{(hyperbolic tangent),} & \coth \phi &= \frac{\cosh \phi}{\sinh \phi} && \text{(hyperbolic cotangent).} \end{aligned}$$

**Problem 3.** Plot the above hyperbolic functions.

**Problem 4** (Pythagorean hyperbolic identity). Prove that  $\cosh^2 \phi - \sinh^2 \phi = 1$ .

**Problem 5.** Express  $\sinh(x \pm y)$  and  $\cosh(x \pm y)$  via  $\sinh x$ ,  $\cosh x$ ,  $\sinh y$  and  $\cosh y$ .

**Problem 6.** Express  $\tanh(x \pm y)$  via  $\tanh x$  and  $\tanh y$ .

**Problem 7.** Inverse hyperbolic functions are denoted by  $\text{Arsinh}$ ,  $\text{Arcosh}$ ,  $\text{Artanh}$  and  $\text{Arctanh}$ . Express  $\text{Arsinh } x$ ,  $\text{Arcosh } x$  and  $\text{Artanh } x$  using the natural logarithm function  $\ln(x)$  and  $x$ .

### 4 Lorentz Transformations

We can now find a relativistic analog for the equation

$$(t', x', y', z') = (t, x + vt, y, z)$$

found in Galilean relativity.

Throughout, we will assume that at time  $t = t' = 0$ , you and Einstein agree on the location of the origin  $(x, y, z) = (x', y', z')$ , and that at all times,  $y' = y$  and  $z' = z$ . Thus we'll just focus on calculating  $(t', x')$  in terms of  $(t, x)$  and vice versa.

Now say that an event happens along the  $x$ -axis, and Einstein measures its coordinates as  $(t, x)$ , while you measure its coordinates as  $(t', x')$ .

**Problem 8.** Based on what you observe at time  $t'$ , find a formula for  $x'$  in terms of  $x$  and  $t'$ . Solve for  $x$  in terms of  $(t', x')$ .

**Problem 9.** When Einstein observes the event at time  $t$ , how far away is the event from the origin in your coordinates, according to his measurements?

Find a formula for  $x'$  in terms of  $t$  and  $x$ .

**Problem 10.** Solve for  $(t', x')$  in terms of  $(t, x)$ .

This transformation, turning  $(t, x)$  into  $(t', x')$  or vice versa, is called a *Lorentz transformation*, after Hendrik Lorentz, who discovered a lot of this before Einstein.

**Problem 11.** Solve for  $(t, x)$  in terms of  $(t', x')$ . Show that this Lorentz transformation is the same as the previous one, but swapping the sign on  $v$ .

**Problem 12.** Show that if  $v$  is much, much smaller than  $c$ , then  $(t', x') \approx (t, x + vt)$ , showing that for very low speeds  $v$ , Einsteinian relativity is basically the same as Galilean relativity.

**Problem 13.** Show that this transformation preserves the quantity  $-(ct)^2 + x^2 + y^2 + z^2$ .

**Problem 14** (Optional: uses matrices). Write the transformation sending  $(t, x, y, z)$  to  $(t', x', y', z')$  as a  $4 \times 4$  matrix.

**Problem 15.** Suppose Einstein sees another train go by on a parallel track at velocity  $u$ . What velocity do you measure this train at?