MATHEMATICS OF JUGGLING

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UCLA MATH CIRCLE ADVANCED 1

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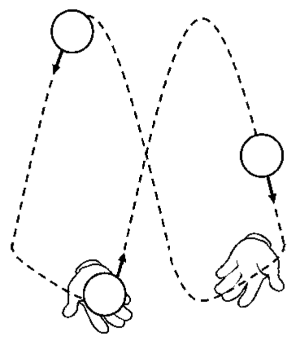
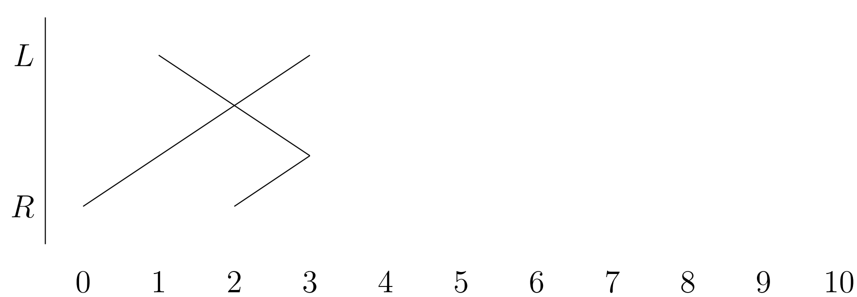
Today, we will talk about juggling. One of the most basic juggling patterns that you may be familiar with is called the *3-ball cascade*. It looks something like this:


[1]

If you're intimidated by this image, that's normal. It's hard to communicate juggling patterns in an image. That's why jugglers have developed special ways to talk about patterns.

Definition 1. The *ladder diagram* of a juggling pattern keeps track of where each ball is throughout time. The x-axis is time, the y-axis is position between the left and right hands. There is one line per ball.

Problem 1. Finish the ladder diagram for the 3-ball cascade, in which at every timestep, a ball is thrown towards the other hand, landing 3 timesteps later.



*Adapted from a lecture by Allen Knutson, which can be found at https://youtu.be/38rf9FLhl-8
[1]Image source: https://www.researchgate.net/figure/Schematic-representation-of-the-3-ball-cascade-juggling-pattern_fig1_51854923

These diagrams communicate the pattern effectively, but are still quite unwieldly to draw. Instead, let's introduce a numerical notation.

Definition 2. A sequence of numbers $a_0, \ldots, a_{n-1}$ models *juggling with vanilla siteswaps* by the following rules.

1. At time $i \in \mathbb{N}$, a ball is thrown in the air for $a_{i \pmod n}$ timesteps.
2. The right hand starts, and the hand that throws the next ball always alternates.
3. Every ball is thrown to the hand that will be throwing when the ball lands.

Such a sequence is *valid* if it is never the case that two balls land at the same time. The *period* of a sequence is n, the number of numbers in the sequence.

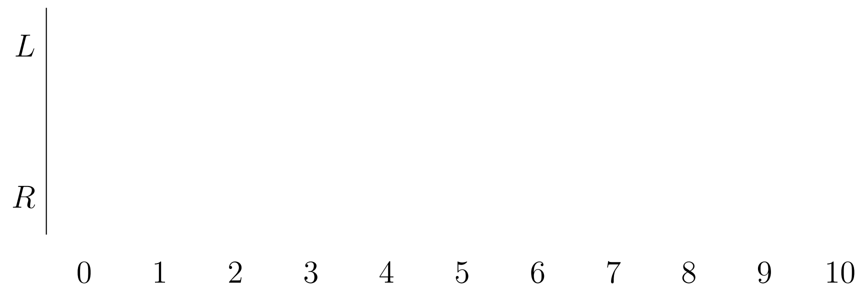
Problem 2. Explain why (3) in the definition above is equivalent to: "throw to the same hand if a_i is even, or opposite hand if a_i is odd."

Problem 3. How can you express the 3-ball cascade in this notation?

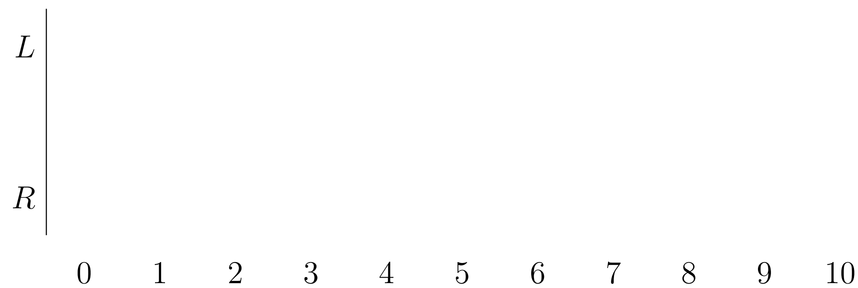
Problem 4. Explain why juggling with vanilla siteswaps is equivalent to *one-handed juggling*, where all balls are thrown and caught by the same hand. (By equivalent, we mean that a sequence of numbers is a valid pattern in one model if and only if it is a valid sequence in the other.)

Problem 5. Are the following ways of throwing balls valid simple juggling patterns? Draw a ladder diagram to figure it out!

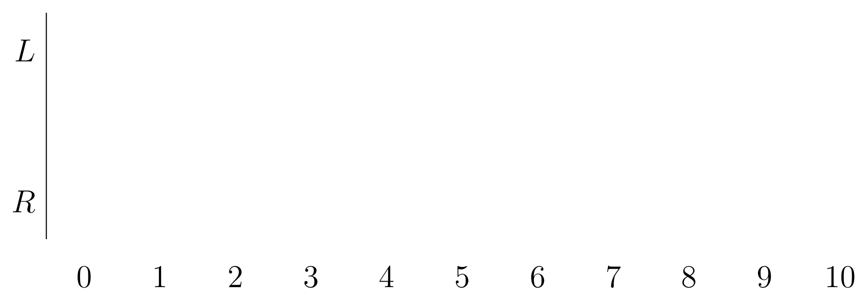
1. 51 (a.k.a. 3-ball shower)



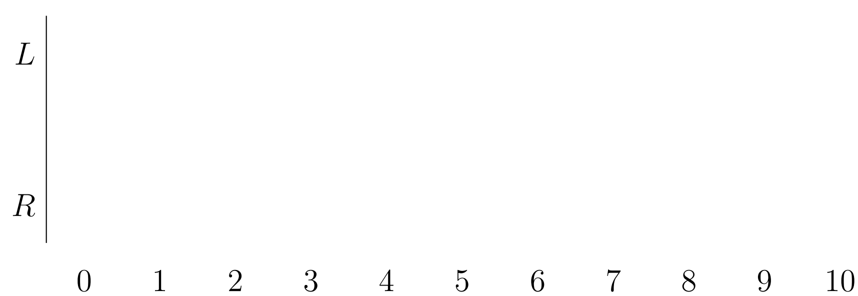
2. 441



3. 324



4. 345



Problem 6. Drawing ladder diagrams is fine, but it can be a pretty slow way to figure out if a pattern is valid. For example, if the pattern is 91, you'd have to draw out at least 9 timesteps to determine validity, even though the period of the pattern is just 2.

Design a method based on calculating something to test if a given sequence of numbers $a_0, \ldots, a_{n-1}$ is a valid juggling pattern with vanilla siteswaps.

(The best method is a linear-time procedure in the period, meaning that a pattern of period n takes something like $an + b$ operations to test, for constants a and b. Don't worry about this at first, but after you find a method, is it linear-time?)

Problem 7. Prove that if $a_0, \ldots, a_{n-1}$ is a valid simple juggling pattern, then the average of $a_0, \ldots, a_{n-1}$ is exactly the number of balls used in the pattern.

This problem has a really creative solution! Think about for a bit, and your instructor has hints if you need them.

Let's switch gears and look at something else. One cool thing that you can do when juggling is to transition from one pattern to another. For example, let's say you're doing a 3-ball cascade (3), and you want to start doing a 3-ball shower (51) instead. Both use 3 balls, so how can you transition between them without starting and stopping?

Definition 3. The *juggling state* after performing throws $a_0, \dots, a_k$ is a sequence of symbols $\times$ and $-$, indicating on which forthcoming timesteps balls will fall on, assuming no more balls are thrown. ($\times$ denotes a falling ball, $-$ denotes a one timestep gap. Do not include trailing $-$'s.)

For example, if we perform throws 5151, the second 1 lands first, followed by the first 5, followed by a gap, and then the second 5. The juggling state is thus $\times\times-\times$.

Problem 8. Determine the juggling state after making the following throws.

1. 333

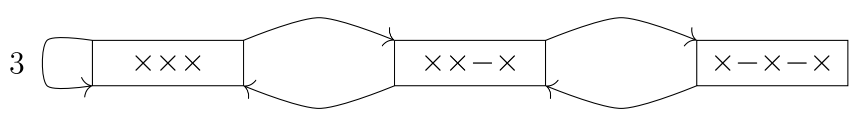
2. 51515

3. 4414

4. 44144

5. 441441

Problem 9. Put a number next to each arrow indicating how many timesteps the next throw should stay in the air in order to change the juggling state from the origin state to the target state. An example is provided. This is called a *transition diagram.* (Hint: try the two arrows on the right first.)



Problem 10. Using the previous problem, describe how to solve the original problem of transitioning between a 3-ball cascade (3) and a 3-ball shower (51).

Problem 11.

1. What are all of the the possible 3-ball juggling states, using throws of at most 5 timesteps?

2. Draw the transition diagram between all of the states from part 1. A part of this diagram should be the same as Problem 9.

3. Using the diagram, what is a valid juggling pattern? Use your transition diagram to come up with some new valid juggling patterns that we haven't considered yet.

Problem 12. (Bonus) Learn the 3-ball cascade! Ask your instructor for some oranges to practice with. Here are some tips:

- Start with 1 orange, then 2 oranges, before 3 oranges. Make sure that you can accurately throw a ball from one hand to another without moving your other hand to catch it.
- Our ladder diagrams represented idealized juggling, in which balls are thrown immediately as they land. Since this hard, you can hold an orange for a beat when you catch it, leading to a ladder diagram like this: