October 2, 2022
AMC 8 Training

Combinatorics

Handout by Sahana Sri

I. What is Combinatorics?

Combinatorics - a mathematical study of counting

Algebra - a mathematical study of relationships

(Calculus - a mathematical of change, area)

II. Factorials

Factorials answer the question "how many ways are there to arrange n objects?" We define this quantity to be $n!$, and it can be calculated as follows:

$$n! = n(n\text{-}1)(n\text{-}2)(n\text{-}3)\ldots 1$$

Some key values you should memorize are:

$5! = 120$

$4! = 24$

$3! = 6$

$2! = 2$

$1! = 1$

$0! = 1$

III. Choose Function

A critical application of factorials is the choose function. The choose function helps us answer the question "how many ways can one choose k objects out of n objects?" This question is subtle, and the answer depends on if the order in which one chooses the objects matters.

If an order doesn't matter, it's a combination.

$$\binom{n}{k} = C_k^n = \frac{n!}{(n-k)!\,k!}$$

Example 1: In how many ways can you choose 3 desserts from a menu of 10 desserts?

Example 2: How many teams of 5 people can you make from a class of 25 students?

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If an order does matter, it's a permutation.

$$P_k^n = \frac{n!}{(n-k)!}$$

Example 1: While listing your 3 favorite deserts from a menu of 10, how many lists can you create?

Example 2: How many ways can you pick a president, vice president, secretary, treasurer, and historian from a class of 25 students?

Mixed Examples:

Example 1.

How many different four-digit numbers can be formed by rearranging the four digits in 2004?

(A) 4 **(B)** 6 **(C)** 16 **(D)** 24 **(E)** 81

AMC 8 2004, Problem 2

Example 2.

How many different combinations of $5 bills and $2 bills can be used to make a total of $17? Order does not matter in this problem.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

AMC 8 2002, Problem 2

Group Work!

Problem 1.

The Centerville Middle School chess team consists of two boys and three girls. A photographer wants to take a picture of the team to appear in the local newspaper. She decides to have them sit in a row with a boy at each end and the three girls in the middle. How many such arrangements are possible?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 12

AMC 8 2015, Problem 4

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Problem 2.

How many 4-digit numbers greater than 1000 are there that use the four digits of 2012?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 12

AMC 8 2012, Problem 10

Problem 3.

How many integers between 1000 and 9999 have four distinct digits?

(A) 3024 **(B)** 4536 **(C)** 5040 **(D)** 6480 **(E)** 6561

AMC 8 2015, Problem 10

Problem 4.

Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen?

(A) 1 **(B)** 3 **(C)** 6 **(D)** 10 **(E)** 12

AMC 8 2004, Problem 17

Problem 5.

Jack wants to bike from his house to Jill's house, which is located three blocks east and two blocks north of Jack's house. After biking each block, Jack can continue either east or north, but he needs to avoid a dangerous intersection one block east and one block north of his house. In how many ways can he reach Jill's house by biking a total of five blocks?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 10

AMC 8 2014, Problem 11

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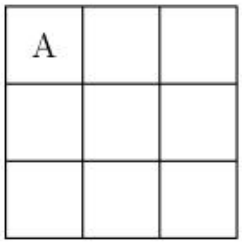
Problem 6.

Three A's, three B's, and three C's are placed in the nine spaces so that each row and column contain one of each letter. If A is placed in the upper left corner, how many arrangements are possible?



(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

AMC 8 2008, Problem 14

Problem 7.

How many 3-digit positive integers have digits whose product equals 24?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 24

AMC 8 2009, Problem 16

Problem 8.

Tyler has entered a buffet line in which he chooses one kind of meat, two different vegetables and one dessert. If the order of food items is not important, how many different meals might he choose?

- Meat: beef, chicken, pork
- Vegetables: baked beans, corn, potatoes, tomatoes
- Dessert: brownies, chocolate cake, chocolate pudding, ice cream

(A) 4 (B) 24 (C) 72 (D) 80 (E) 144

AMC 8 2001, Problem 14

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Problem 9.

Bicycle license plates in Flatville each contain three letters. The first is chosen from the set {C,H,L,P,R}, the second from {A,I,O}, and the third from {D,M,N,T}.

When Flatville needed more license plates, they added two new letters. The new letters may both be added to one set or one letter may be added to one set and one to another set. What is the largest possible number of ADDITIONAL license plates that can be made by adding two letters?

(A) 24 (B) 30 (C) 36 (D) 40 (E) 60

AMC 8 1999, Problem 15

Problem 10.

The "Middle School Eight" basketball conference has 8 teams. Every season, each team plays every other conference team twice (home and away), and each team also plays 4 games against non-conference opponents. What is the total number of games in a season involving the "Middle School Eight" teams?

(A) 60 **(B)** 88 **(C)** 96 **(D)** 144 **(E)** 160

AMC 8 2014, Problem 16

Problem 11.

Ali, Bonnie, Carlo, and Dianna are going to drive together to a nearby theme park. The car they are using has 4 seats: 1 driver's seat, 1 front passenger seat, and 2 back passenger seats. Bonnie and Carlo are the only ones who know how to drive the car. How many possible seating arrangements are there?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 12 **(E)** 24

AMC 8 2003, Problem 16

Problem 12.

The Little Twelve Basketball League has two divisions, with six teams in each division. Each team plays each of the other teams in its own division twice and every team in the other division once. How many games are scheduled?

(A) 80 **(B)** 96 **(C)** 100 **(D)** 108 **(E)** 192

AMC 8 2005, Problem 14

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Problem 13.

Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 9 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 18$

AMC 8 2013, Problem 21

Problem 14.

Let R be a set of nine distinct integers. Six of the elements are 2, 3, 4, 6, 9, and 14. What is the number of possible values of the median of R?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

AMC 8 2012, Problem 22

Problem 15.

How many 4-digit positive integers have four different digits, where the leading digit is not zero, the integer is a multiple of 5, and 5 is the largest digit?

$\textbf{(A)}\ 24 \qquad \textbf{(B)}\ 48 \qquad \textbf{(C)}\ 60 \qquad \textbf{(D)}\ 84 \qquad \textbf{(E)}\ 108$

AMC 8 2011, Problem 23

Problem 16.

A baseball league consists of two four-team divisions. Each team plays every other team in its division N games. Each team plays every team in the other division M games with $N > 2M$ and $M > 4$. Each team plays a 76-game schedule. How many games does a team play within its own division?

$\textbf{(A)}\ 36 \qquad \textbf{(B)}\ 48 \qquad \textbf{(C)}\ 54 \qquad \textbf{(D)}\ 60 \qquad \textbf{(E)}\ 72$

AMC 8 2015, Problem 24