Logic I

Kevin Li

September 2022

1 Review

Last year, we introduced various definitions and problems in *binary arithmetic* as an application of logic. This quarter will be focused on a slightly more abstract topic: **Propositional Logic**. This generally includes working with deductions in compound statements rather than arithmetic. Recall a few useful definitions:

Definition 1. *A* ***statement*** *is an expression which is either True or False.*

Definition 2. ***Logical connectives*** *are the five following symbols:* $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$

Definition 3. ***Quantifiers*** *are the two following symbols:* $\exists$, $\forall$

Problem 1. *a) Write down two sentences that are statements and two sentences which are not statements*

b) State the meaning of each of the 7 symbols in definitions 2 and 3.

Problem 2. *Fill out the following truth table (T = True, F = False):*

A	B	$\neg A$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$A \Leftrightarrow B$	$A \vee \neg A$	$\neg A \vee B$	$(A \Rightarrow B) \Rightarrow A$
T	T								
F	T								
T	F								
F	F								

2 Propositional Logic

Boolean Algebra and Propositional Logic are similar, in the sense that we are trying to deduce what statements are true. Previously, we saw rules of addition and multiplication. However, propositional logic is a bit more intuitive, using True (T) and False (F) instead. To be more precise, we will also be using two more symbols: '(' and ')'.

Problem 3. *For the following, assign either (T) or (F) to each of the statements A,B,C to make the compound statement True.*

a) $((\neg A \Rightarrow (\neg B)) \Rightarrow (A \wedge B))$

b) $(\neg A \vee B) \wedge (B \vee \neg C) \wedge (A \vee C)$

c) $(((A \vee B) \Rightarrow (\neg C \wedge \neg B)) \wedge (\neg B \Rightarrow C))$

Definition 4. *A* ***Propositional Language*** $\mathcal{L}$ *in propositional logic is a nonempty set of statements (we call such statements* ***Atomic Propositions****)*

Definition 5. *The* ***sentences*** *of* $\mathcal{L}$ *are defined as follows:*

1. *Every element of* $\mathcal{L}$ *is a sentence*

2. *If* φ *is a sentence of* $\mathcal{L}$ *then so is* $(\neg\varphi)$

3. *If* φ *and* ψ *are sentences of* $\mathcal{L}$ *then* $(\varphi \vee \psi), (\varphi \wedge \psi), (\varphi \Rightarrow \psi), (\varphi \Leftrightarrow \psi)$ *are sentences of* $\mathcal{L}$

4. *Any finite sequence of symbols is a sentence of* $\mathcal{L}$ *only if they can be created by a finite number of applications of 1,2,3*

Problem 4. *Let* $\mathcal{L} = \{A_1, A_2, A_3\}$ *propositional language. Determine which of the following are sentences:*

A_3 $\qquad$ $(A_1 \vee A_2)$ $\qquad$ $(\neg(A_1 \wedge (A_2 \vee A_2)))$ $\qquad$ $\neg A_1)$

Problem 5. *Let $\mathcal{L}$ be a propositional language. Prove for any sentence φ of $\mathcal{L}$, the number of left parenthesis occurring in φ is equal to the number of right parenthesis occurring in φ. (Hint: Induction. Create functions $L(\varphi), R(\varphi)$)*

3 Working with quantifiers

In your classes and various Math Circle worksheets, quantifiers in particular will become very important. For example, the following is a important definition for calculus and analysis.

Definition 6 (continuous)**.** *A function* $f : \mathbb{R} \to \mathbb{R}$ *is* continuous *if for all* $x \in \mathbb{R}$ *and all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for all* $y \in \mathbb{R}$ *satisfying* $|x - y| < \delta$*, we have* $|f(x) - f(y)| < \epsilon$*.*

These are definitely confusing for everyone at first! This section is dedicated to helping you feel more confident working with these kinds of expressions.

Problem 6. *Negate the following statements.*

0. *For all* $x \in \mathbb{N}$*, there exists* $y \in \mathbb{N}$ *such that* $x = y$*.*

   Example solution. There exists $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$, we have $x \neq y$. (Do you see the general strategy? Do you see why it works?) □

1. *There exists* $x \in \mathbb{N}$ *such that for all* $y \in \mathbb{N}$*, we have* $2y < x$*.*

2. *For all* $a \in \mathbb{R}$*, there exists* $b \in \mathbb{R}$ *such that for all* $c \in \mathbb{R}$*, there exists* $d \in \mathbb{R}$ *such that* $ab = cd$*.*

Problem 7. *True or False. If true, prove that the statement is true. If false, prove that the statement is false (i.e. prove that the negation is true).*

1. *There exists* $x \in \mathbb{N}$ *such that* $x > 5$*.*

2. *For all* $x \in \mathbb{N}$*, we have* $x > 5$*.*

3. *There exists* $x \in \mathbb{N}$ *such that for all* $y \in \mathbb{N}$*, we have* $2y < x$*.*

4. *For all* $y \in \mathbb{N}$*, there exists* $x \in \mathbb{N}$ *such that* $2y < x$*.*

5. *There exists* $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$ *such that for all* $z \in \mathbb{R}$*, we have* $xz^2 < yz^4$*.*

6. *For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$ with $z > 0$, we have $x + y < z$.*

Problem 8.

1. *Prove that $f(x) = x$ is continuous, directly from the above definition. (Hint: choose $\delta = \epsilon$. Do not worry about what things "mean" for now, just follow the same pattern as the previous problem.)*

2. *Explain how each part of the definition of continuity relates the following informal explanation: "A function is continuous if around any point, you can make the outputs fit in a small interval by taking the inputs in a sufficiently small interval."*

3. *Explain how part (b) relates to the even more informal explanation: "A function is continuous if you can draw its graph without lifting your pen." Why do we care about part (a) (i.e. formal proofs) if we also have such a simple explanation?*

4. *Prove that the piecewise defined function*

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is not continuous. (Hint: first negate the definition of continuous.)

Problem 9. *Sometimes, authors will confusingly write quantifiers in a different order. Put the following statements back in standard math order (all quantifiers first, statement at the end). (This is often useful when you want to negate the statement.)*

1. $a^2 + b^2 = c^2$, *where* a, b, c *are the sides of any right triangle.*

2. *There exists* $e \in G$ *such that* $x \cdot e = x$ *for all* $x \in G$.

3. *For all* $x \in \mathbb{R}$, *we have* $x \cdot y = 1$ *for some* $y \in \mathbb{R}$.