Combinatorial Game Theory and Nim

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Ice-breaker.

We begin today by playing an ice-breaker game. Everyone will guess a number $0 \leq x \leq 100$. Let's say there are n students, and each student guesses x_i. Then, the instructors will compute:

$$T = \frac{2}{3} \cdot \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{2}{3} \cdot \bar{x}.$$

Notice that T is $2/3$ of the class average $\bar{x}$. Your score is $|T - x|$. The winner will be the student with the smallest score; essentially, you want your guess to be as close to T as possible. If there are multiple students with the least score, all of them will be winners. The winner(s) will win a prize!

Introduction.

Today, we will be playing a game called Nim. It is a strategic game in which players take turns removing toothpicks from distinct piles. The goal of the game is to be the player who removes the last toothpick from the game. Here is how the game is played.

There will be k piles, enumerated as $p_1, p_2, \cdots, p_k$, where pile p_i has n_i toothpicks in it at any given time. A turn consists of a player removing $1 \leq n \leq n_i$ toothpicks from pile p_i for some $1 \leq i \leq k$. The two players of the game will take turns until the last toothpick is taken; the player who takes the last toothpick is the winner.

Exercise 1. *Play a few games of Nim with the people sitting next to you. Try to develop a strategy!*

Strategies.

Now, we're looking for ways that we can guarantee a win.

Exercise 2. *Prove that Player 1 always wins one-pile Nim (hint: don't overthink it!).*

Exercise 3. *Prove that given two piles p_1 and p_2 of size $n_1 = 3$ and $n_2 = 4$ respectively, Player 1 always has a winning strategy.*

Exercise 4. *Prove that given two piles p_1 and p_2 of size $n_1 = 4$ and $n_2 = 4$ respectively, Player 2 always has a winning strategy.*

Exercise 5. *Assume there is one pile p_1 with size $n_1 > 0$ and another pile p_2 with size $n_2 = 1$. For what values of n_1 does Player 1 have a winning strategy? How about Player 2?*

Binary Representation and XOR.

Typically, the numbers we see are in decimal representation. This mean when we see a number like 12345, it can be written as:

$$12345 = 1 \cdot 10^4 + 2 \cdot 10^3 + 3 \cdot 10^2 + 4 \cdot 10^1 + 5 \cdot 10^0 = 10000 + 2000 + 300 + 40 + 5.$$

As we can observe, the kth position to the left of the decimal represents the coefficient (or multiplier) on the 10^k term. We can view 12345 as 1 ten-thousand, 2 thousands, 3 hundreds, 4 tens, and 5 ones. In general, with $d_i \in \{0, 1, 2, \cdots, 9\}$ (read as d_i are in the set $\{0, 1, 2, \cdots, 9\}$) we can write any number x as:

$$x = d_0 \cdot 10^0 + d_1 \cdot 10^1 + \cdots + d_n \cdot 10^n.$$

Then, the decimal representation of x is $d_n \cdots d_1 d_0$. Similarly, we can express any positive integer in base-2 or binary as with the digits 0 or 1 representing how many powers of 2 a number is comprised of. For instance, the number 110101 in binary can be expressed in decimal as:

$$1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 32 + 16 + 0 + 4 + 0 + 1 = 53.$$

In general (you don't need to prove this), with $b_i \in \{0,1\}$ (read as b_i are in the set $\{0,1\}$) we can write any number x as:

$$x = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \cdots + b_n \cdot 2^n.$$

Then, this number can be represented as $b_n \cdots b_1 b_0$ in binary.

Exercise 6. *Compute the binary representation of 103.*

Exercise 7. *What number has binary representation 1101? Give your answer in decimal.*

Exercise 8. *This exercise will ask you to come up with a divisibility rule. Given a number b in binary, can you tell when b is divisible by four?*

Now, we are going to define an operation $\otimes$. Given two "bits" $x, y \in \{0,1\}$ (read as x, y are in the set

$\{0,1\}$) we define $x \otimes y$ as follows:

$$0 \otimes 0 = 0$$

$$0 \otimes 1 = 1$$

$$1 \otimes 0 = 1$$

$$1 \otimes 1 = 0.$$

This operation $\otimes$ is called XOR, or "exclusive or" (why?). Now, given two binary numbers a and b, define $a \otimes b$ as the digit-wise XOR of a and b. For example:

$$0000 \otimes 0000 = 0000$$

$$0000 \otimes 1111 = 1111$$

$$0110 \otimes 1101 = 1011$$

$$1011 \otimes 0010 = 1001.$$

Exercise 9. *Compute* $(1101 \otimes 1011) \otimes 0010$ *and* $1101 \otimes (1011 \otimes 0010)$*. Prove that XOR is associative; namely, prove that* $(a \otimes b) \otimes c = a \otimes (b \otimes c)$*.*

Exercise 10. *Compute* $1101101 \otimes 0010101$ *and* $0010101 \otimes 1101101$*. Prove that XOR is commutative; namely, prove that* $a \otimes b = b \otimes a$*.*

The above two exercises show that we don't need parentheses when using XOR, and the order of our inputs doesn't matter!

Solving Nim.

Define the nim-sum of a game of Nim as follows. Write $n_1, n_2, \cdots, n_k$ in binary as $b_1, b_2, \cdots, b_k$. Then, compute $S = b_1 \otimes b_2 \otimes \cdots b_k$, where $\otimes$ is the XOR operation defined above (recall that parentheses and order don't matter here). Now, the nim-sum is the number represented by the binary string S.

Exercise 11. *Prove that the end-state (with no toothpicks remaining) has zero nim-sum.*

Exercise 12. *Prove that given a game of Nim with non-zero nim-sum, we can make one move to get to a game of Nim with zero nim-sum.*

Exercise 13. *Prove that given a game of Nim with zero nim-sum, any move will take us to a game of Nim with non-zero nim-sum.*

Exercise 14. *Suppose we have a game of Nim, and you know the nim-sum. Using the above exercises, can you tell who has a guaranteed winning strategy?*

Great! You now know how to win Nim, and you can impress your friends.

Challenge Problems.

Exercise 15. *This exercise will ask you to come up with a divisibility rule. Given a number b in binary, can you tell when b is divisible by three?*

Exercise 16. *Prove that XOR has an identity element; namely, show that there is a binary number e such that $e \otimes y = y$ for all other binary numbers y and show that it is unique. Also, show that given some binary number x there is another binary number x' such that $x \otimes x' = e$ and show that this number x' is unique. This (along with the above exercises) means that the set of binary strings together with the XOR operation, forms an "abelian group"!*

Exercise 17. *You find yourself in Shannon's Prison for Unruly Children :(. You and a prison buddy of yours, SJ, love playing logic and math puzzles with each other. The warden of the prison takes note of SJ and your abilities and offers to let you play a game to escape the prison. Here is how it is played. An* 8×8 *chessboard has a coin either heads of tails up on each of its 64 positions. The warden hides a key under exactly one position of the chessboard. You will be shown where this key is hidden and you'll be allowed to flip exactly one coin. Your pal, SJ, will come into the room and observe the chessboard, without interacting with you after you've made the flip, and must deduce the position of the key on the board. Can you collaborate with SJ to come up with a strategy before playing the game to ensure that you will win and be freed from Shannon's Prison for Unruly Children.*