ORMC AMC Group: Week 1
AMC Basics

September 24, 2022

1 Algebraic Manipulations

Some useful factoring identities:

- $(a+b)^2 = a^2 + 2ab + b^2$
- $a^2 - b^2 = (a-b)(a+b)$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} - a^{2n-1}b + \cdots + b^{2n})$
- $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$
- $(a+1)(b+1) = ab + a + b + 1$

1.1 Problems

1. Given real numbers a, b with $a + b = 9$ and $ab = 7$, find $a^3 + b^3$.

2. Given $3x + \dfrac{1}{2x} = 3$, find $8x^3 + \dfrac{1}{27x^3}$.

3. **(Harder)** Find all pairs of positive integers (a, b) with $a \leq b$ which satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$.

4. **(Harder)** Evaluate $2022^3 - 2022^2 \cdot 2023 - 2022 \cdot 2023^2 + 2023^3$.

2 Equation Techniques

- Isolation: get target variable by itself (single equation)
  - i.e. $x + 2 = 2x \implies 2 = x$
  - [subtract x from both sides to isolate]
- Substitution: get one variable in terms of other(s), and substitute. (system of equations)
  - i.e. $x + y = 4x,\ y = 5x^2 + 2 \implies y = 3x,\ 3x = 5x^2 + 2$
  - [solve for y, substitute $3x$ for y]
- Elimination: add/subtract/multiply/etc equations to eliminate a variable. (system of equations)
  - i.e. $x + y = 4x,\ y = 5x^2 + 2 \implies x + y = 4x,\ -x = 5x^2 + 2 - 4x$
  - [subtract first equation from second equation]

2.1 Problems

1. Find all x such that
$$\frac{1}{\sqrt{x-5}+1} + 3 = \frac{4}{\sqrt{x-5}+1}$$

2. Find all pairs (x, y) such that
$$a(a - 7b) = 60,$$
$$b(7b - a) = 20.$$

3. Suppose that x, y, z are real numbers such that
$$x = y + z + 2,$$
$$y = z + x + 1,$$
$$z = x + y + 4.$$
Compute $x + y + z$.

4. Solve the following system of equations:
$$2x_1 + x_2 + x_3 + x_4 = 1,$$
$$x_1 + 2x_2 + x_3 + x_4 = 2,$$
$$x_1 + x_2 + 2x_3 + x_4 = 3,$$
$$x_1 + x_2 + x_3 + 2x_4 = 4$$

3 Functions

A function is an expression f that takes an input x, and outputs at most one value $y = f(x)$. Functions may take multiple inputs, like $f(x, y, z)$. In this case, there can be at most one output per ordered tuple (x, y, z).

1. **domain**: The input set X of values x that correspond to at least one output $f(x)$.
2. **codomain**: The set Y of values that $f(x)$ might possibly output on an arbitrary input x.
3. **range**: A subset of the codomain, consisting of all values $f(x)$ which are outputs of the function for some input x in the domain. When the domain is X, the range is written as $f(X)$.

Example: $f : \mathbb{Z} \mapsto \mathbb{Z}$ defined by $f(x) = |x|$ would have domain and codomain $\mathbb{Z}$, and range $\mathbb{N} \cup \{0\}$.

Also important is **composition**, applying one function to the output of another to create a new function, i.e. $h(x) = f(g(x))$. Often written as $f \circ g(x)$.

Functions are often defined explicitly, like $f(x) = \frac{1}{1-x-x^2}$. They may also be defined **recursively**, meaning that it is defined in terms of other outputs, and some base case(s). If you're familiar with inductive proofs, the idea is similar.

For example, we can define a function f on the integers as:

$$f(0) = 0,\ f(1) = 1, \text{ and } f(n+2) = f(n+1) + f(n) \text{ for } n \geq 0.$$

3.1 General Function Problems

1. A function is defined recursively by
$$f(1) = f(2) = 1,$$
$$f(n) = f(n-1) - f(n-2) + n,\ n \geq 3.$$
Find $f(2018)$.

2. If $\sum_{n=0}^{2020} f(n) = 0$, and $f(n+1) = \dfrac{f(n)-1}{f(n)+1}$, find $f(0)$.

3. If $f(x) = \dfrac{x}{x+1}$, what is $f(f(f(f(2009))))$?

3.2 Function Inverses

A function g is the **inverse function** of f when:

- $g(f(x)) = x$ for all x in the domain of f.
- $f(g(x)) = x$ for all x in the domain of g.

When this is the case, we may write g as f^{-1}, indicating that it is the inverse of f. Note that the definition is symmetric, so f is also the inverse of g, and we may write f as g^{-1}.

3.3 Logarithms and Exponents

An important subset of functions and inverses is exponential functions and their inverse, logarithms. Given an equation $a^b = c$, if we know the values of a and c, we may use a logarithm to find b.
We define $\log_b(x) = y$ precisely when $b^y = x$. We call b the **base** of the logarithm. When the base is omitted, it is usually implied to be 10.

Two important properties that follow immediately from this definition are:

$$\log_b(b^x) = x, \quad b^{\log_b(x)} = x$$

Two other important properties are:

- $\log_b(x) + \log_b(y) = \log_b(x \cdot y)$
  - Proof: $b^{\log_b(x)+\log_b(y)} = b^{\log_b(x)} \cdot b^{\log_b(y)} = x \cdot y = b^{\log_b(x \cdot y)} \implies \log_b(x) + \log_b(y) = \log_b(x \cdot y)$.
- **change-of-base**: $\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$
  - Proof: $x = c^{\log_c(x)} = b^{\log_b(x)} = (c^{\log_c(b)})^{\log_b(x)} = c^{\log_c(b) \cdot \log_b(x)} \implies \log_b(x) = \frac{\log_c(x)}{\log_c(b)}$

3.3.1 Logarithm Exercises

1. Similar to the two proofs above, show the following three properties of logarithms:
   - $\log_b(x) - \log_b(y) = \log_b(x/y)$
   - $\log_b(x^y) = y \cdot \log_b(x)$
   - $\log_b(1) = 0$
2. Which of the following is the value of $\sqrt{\log_2(6) + \log_3(6)}$?
   (a) 1
   (b) $\sqrt{\log_5(6)}$
   (c) 2
   (d) $\sqrt{\log_2(3)} + \sqrt{\log_3(2)}$
   (e) $\sqrt{\log_2(6)} + \sqrt{\log_3(6)}$