Special Relativity

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1 Galilean Relativity

It's the early 1600s, in Italy. Your friend Galileo Galilei wants to make a point about motion. He's gone out into the harbor on a boat.

Let t, x, y, z be the time and space coordinates Galileo measures on the moving boat. Let t', x', y', z' be the time and space coordinates you measure, watching his experiment. You've synchronized your watches at the same time $t = t' = 0$, the boat is moving at velocity v in the x-direction, which is parallel to the shore, and the z-direction is straight up. At time 0, you both agree that the bottom of the mast of the boat is at the origin $(0, 0, 0)$.

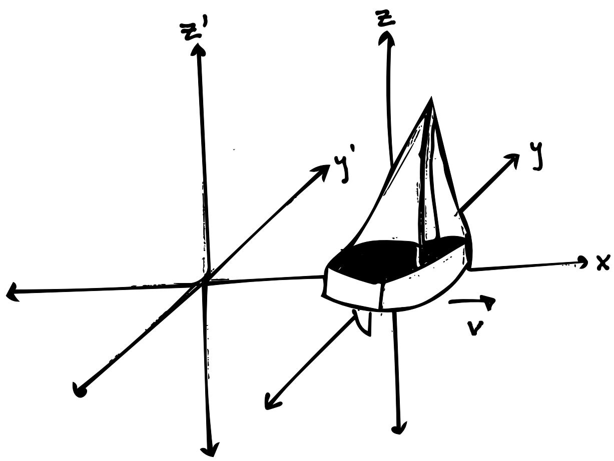

Figure 1: The boat and the two coordinate systems

Problem 1. Galileo has climbed up to the top of the mast, and drops a ball down to the deck at $t = 0$, letting it fall straight down.

You both measure that it takes the same time, T, for the ball to hit the deck, and you both see it hit the same spot on the deck - right at the bottom of the mast.

What are the coordinates (t, x, y, z) of the ball hitting the boat, by Galileo's measurements?

Solution. He measures $(T, 0, 0, 0)$ - it reaches the origin at time T.

*Drawings by Nadine Bradbury

Problem 2. What are the coordinates (t', x', y', z') of the ball hitting the boat, by your measurements on the shore?

> **Solution.** You measure $(T, vT, 0, 0)$, as the time is still T, but the origin has moved vT further along the x-axis.

Problem 3. In general, if Galileo measures something happening at time t and position (x, y, z), what are the coordinates (t', x', y', z') you measure?

> **Solution.** In general, $(t', x', y', z') = (t, x + vt, y, z)$, as you will agree on measurements of time and the y, z-axes, but the boat will have moved a distance vt along the x-axis, adding vt to his x-measurement.

Problem 4. Say that an object on the boat is moving at velocity $\bar{v} = (v_x, v_y, v_z)$ if its velocity in the x-direction is v_x, its velocity in the y-direction is v_y, and its velocity in the z-direction is v_z, per Galileo's measurements. What is the velocity $\bar{v}' = (v'_x, v'_y, v'_z)$ that you measure?

> **Solution.** You will measure $\bar{v}' = \bar{v} + (v, 0, 0)$, as the x-velocity will be increased by v, and other velocities will be unchanged.

2 Time Dilation

It's 1905, in Switzerland. You're standing on the platform of a train station, while your friend Albert Einstein is standing on a train moving along the x-axis at velocity v, holding a device called a *light clock*. Recent experiments have shown that the speed of light is a physical constant, that everyone will measure the same - let's call this c. We will see the consequences of this.

As in the case of the boat, the z-direction is up, and the y-direction is perpendicular to the train tracks.

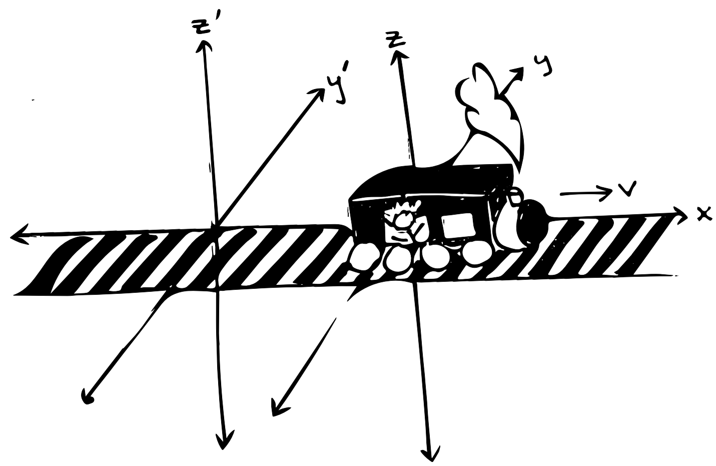

Figure 2: The train and the two coordinate systems

In the light clock, a beam of light is sent up distance L and bounced off a mirror down to a detector at the point it started from. Assume that in Einstein's coordinates (t, x, y, z), and yours (t', x', y', z'), the light leaves the detector at $(0, 0, 0, 0)$.

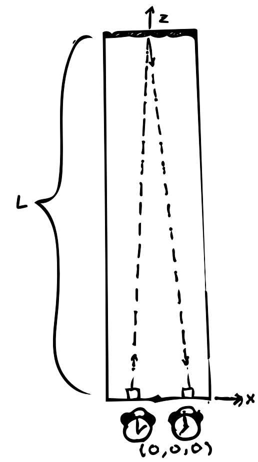

Figure 3: The light clock from Einstein's perspective

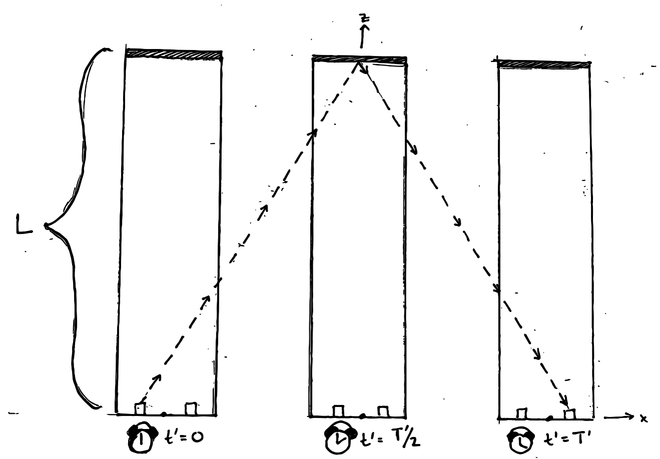

Figure 4: The light clock from the perspective of the platform

Problem 5. Let T be the time Einstein measures when the light returns to the detector, and let T' be the time that you measure when light returns to the detector. If you both measure the same constant speed of light c, then what is $\frac{T'}{T}$? Let us call this ratio γ.

(Hint: Measure the distance the light travels, from each perspective, to solve for the amount of time.)

Solution. The distance Einstein measures the light moving at should be cT, and the distance you measure should be cT'.

The distance Einstein measures as the light goes up and down is just $2L$. To you, as the light goes up and down, it also moves vT' to the right, so the distance is $2\sqrt{(vT'/2)^2 + L^2} =$

$\sqrt{(vT')^2 + (2L)^2} = cT'$. We thus find that

$$(c^2 - v^2)T'^2 = (2L)^2 = (cT)^2,$$

so

$$\gamma = \frac{T'}{T} = \frac{c}{\sqrt{c^2 - v^2}} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

This may seem incredibly counter-intuitive, as in our normal world, we assume everyone perceives time the same way, but we now find that time seems to be moving at different speeds (this is called *time dilation*). Let's see how much of an effect this would have on normal daily life:

Problem 6. Say that the train is moving incredibly fast, at say $c/2$. How big is γ, and who perceives the light clock as ticking faster, you or Einstein?

Solution. We find $\gamma = \frac{1}{\sqrt{1-1/4}} \approx 1.15$. This is greater than 1, so you will find that the light clock is ticking 15% more slowly.

The speed of light c is approximately $300,000,000$ meters per second. The exact value is $299,792,458$ meters per second. It may be called exact since in 1983 the 17th meeting of the General Conference on Weights and Measures redefined the meter as "the length of the path traveled by light in vacuum during a time interval of 1/299792458 of a second".

Problem 7. A possible verification of the special theory of relativity involves explicitly measuring the phenomenon of time dilation. To do this, we put a precise atomic clock on an airliner flying at a constant speed of 800 km/h. Another identical atomic clock is left on the ground and serves as an observer. Then we compare the times (easier said than done!) on the two clocks at the end of a trip.

Most timepieces people use to tell time are accurate to within 10 or 15 seconds every month. Fancy mechanical watches (like a Rolex) will be off by more — a second or two each day. What must be the minimum precision of the atomic clocks to highlight the time dilation phenomenon?

Solution. This would be $\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\frac{800*1000/3600}{300000000}^2}} \approx 1.000000000000274$. An absolutely tiny difference. It is approximately 8 microseconds per year.

A light-year is the distance that light travels in a vacuum in one Julian year (365.25 days). It is equivalent to about 9.46 trillion kilometers.

Problem 8. The Andromeda Galaxy is located 2.3 million light-years from Earth. We consider the Earth and Andromeda to be points with no relative motion. We decide to visit Andromeda thanks to a rocket traveling at a constant speed from Earth to Andromeda.

1. If the time taken to make the trip is 2.31 million years for terrestrial observers, what is the rocket's speed? What is the duration of travel for rocket passengers? Will they still be alive at the end of the expedition? What about their descendants?

2. Answer the same questions if we embark on a more powerful rocket that only takes 2.301 million years to make the travel.

3. We want the rocket expedition not to exceed 20 years for the passengers. What is the minimum speed the rocket needs to travel at to carry out this project? How long does this trip take for a terrestrial observer?

Solution. The rocket speed is $\frac{230}{231}c$. Then $\gamma = \frac{1}{\sqrt{1-(v/c)^2}} \approx 10$. So the travel would appear 230 thousand years to the passengers.

If the rocket speed is $\frac{2300}{2301}c$, then $\gamma = \frac{1}{\sqrt{1-(v/c)^2}} \approx 34$. So the travel would appear 67 thousand years to the passengers.

For the travel to last 20 years for passengers, γ should be around 115 000. That means $v \approx 0.99999999996219\ c$. The time travel would take is 2300000.0000869 years or 2.3 million years and 46 minutes.