Week 8: Ramsey Theory

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1 Friends and Anti-Friends

Definition 1. *A graph is* 2-colored *if each edge is assigned one of two different colors. For this handout we will refer to the colors as red and blue, but if you don't have any colored pens/pencils, you can draw them as solid and dashed lines.*

Problem 1. *How many ways can you 2-color the edges of a K_3 graph? How about a K_5 graph? How about K_n, for any n?*

Let us call two people friends if they know each other. Let us call them strangers otherwise.

Definition 2. *We'll call a set of people a* friend group *if every person in the set is friends with every other person in the set. A set of people is an* anti-friend group *if all pairs of people in the set are strangers.*

Problem 2. *Can you create a set of 4 people such that none of them form a 3 person friend group or anti-friend group? How about with 5 people? (Show your answers with a 2-colored graph)*

Problem 3. *Try to create a set of 6 people such that none of them form a 3-person friend group or anti-friend group.*

Problem 4. *Prove that, among any six people, there is either a 3-person friend group or a 3-person anti-friend group.*

Definition 3. *The* Ramsey Number $R(m, n)$ *is the minimum number of people that must exist to guarantee a friend group of size* m *or an anti-friend group of size* n.

Problem 5. *What is* $R(3, 3)$*?*

Problem 6. *Find* $R(2, 5)$.

Problem 7. *Is* $R(m, n) = R(n, m)$ *for any* m, n*? Why or why not?*

Theorem 1. (Ramsey) *The number $R(m,n)$ exists for any positive integers m,n.*

Frank Plumpton Ramsey (1903-1930), a British philosopher, mathematician, and economist.

Philosophical observation: pick up two natural numbers r and b. Take $R(r,b)$ or more points. Connect each of the points to all others, choosing one of the two different colors, red or blue, at random. What you get seems to be totally chaotic. However, one will always be able to find a monochromatic subgraph, either a red one with r vertices or a blue one with b vertices, within the original graph. Chaos generates order!

2 More Ramsey Theory

Problem 8. *Suppose there are 12 people in a room. Prove there must be at least two 3-person friend groups or anti-friend groups. (Not necessarily of the same type).*

Challenge Problem 9. *Prove that, amongst any 12 people, there must be at least two 3-person friend groups or anti-friend groups* of the same type. *(Hint: try to find as many monochromatic triangles as you can, and then later worry about finding two of the same color among them.)*

Now let's take a look at larger friend groups.

Challenge Problem 10. *Can you show that any group of 27 people must contain a 4-person friend group or anti-friend group? (Hint: run the argument for problem 4 twice in a row.)*

Challenge Problem 11. *Show that any group of* 4^m *people must contain an m-person friend group or anti-friend group.*

The last time we've checked, it was proven that $43 \leq R(5,5) \leq 48$, but the exact value was not known.

Challenge Problem 12. *Find* $R(5,5)$.

3 (Challenge) Utilizing Randomness

Ramsey theory was the motivation for the discovery of another very strange phenomenon in mathematics. This one doesn't have a standard slogan, but here's an attempt: "if you're trying to do something complicated, sometimes you can succeed by doing it randomly".

This phenomenon was discovered when people were trying to solve the anti-Ramsey problem. That is, instead of trying to find a group of 3 friends or anti-friends, they were trying to come up with an arrangement without a group of 3 friends or anti-friends. We saw that this can be done for 5 people but not for 6; that is, there is a way for 5 people to know each other so that no 3 are all friends or all strangers, but the same is not true for 6 people. Well, what about a group of 4 friends or anti-friends? We saw that any 27 people has at least one group of 4 friends or anti-friends, but can we find an arrangement of 26 people that avoids such a grouping?

Problem 13. *In a group of 11 people, how many possible arrangements of friendships are there?*

Problem 14. *Suppose the 11 people are labelled A through K. How many possible arrangements of friendships exist between the 11 people such that A, B, C, D and E form a friend or anti-friend group with each other? What fraction of all possibilities is that?*

Problem 15. *In what fraction of arrangements are D, E, F, G and H all either friends or strangers with each other? In what fraction are either A, B, C, D and E, or D, E, F, G and H (or both) a 5 person friend or anti-friend group? You don't have to find it exactly, an overestimate is enough.*

Problem 16. *How many groups of 5 people exist in a group of 11 people?*

Problem 17. *In the spirit of problem 15, give an overestimate for the fraction of arrangements of 11 people where some group of 5 people are all friends or anti-friends.*

Problem 18. *Using problem 17, show that there is some arrangement of 11 people where no group of 5 people are all friends or anti-friends.*