PROOF, CONTRAPOSITIVE, CONTRADICTION

MATH CIRCLE (BEGINNERS) 04/29/2012

$P \implies Q$ means "If P then Q." For example when P is the statement "Annika drives a car", and Q is the statement "Annika has a driver's license", then $P \implies Q$ is the statement "If Annika drives a car, then she has a driver's license." This kind of statement is called an implication, or a conditional statement.

We will use $\neg P$ to represent the statement "P is false." With P and Q as above, $\neg P$ represents the statement "Annika does not have a driver's license."

For each of the following statements, write down the statement which is its logical negation:

(a) "Clint is the worst Math Circle instructor."

(b) "There exists an even number greater than or equal to 4 which is not the sum of two prime numbers."

(c) "All swans are white."

(d) "London is the capital of England, and Sydney is the capital of Australia."

(e) "These aren't the droids you're looking for."

(f) “Either I’m going crazy, or there’s an elephant floating in mid-air in the other corner of the room.”

(g) “Behind every great man, there’s a great woman.”

(h) “If $x < 10$, then $x^2 < 100$.”

(i) “If it’s not raining, Angela did not bring an umbrella.”

(1) (Pizza) The statement “If Zhubeen eats 50 pieces of pizza in the morning, he will feel very sick in the afternoon.” is a true statement.

Which of the following is logically equivalent to the above statement?

(a) If Zhubeen does not eat 50 pieces of pizza in the morning, he will not feel very sick in the afternoon.

(b) If Zhubeen does not feel very sick in the afternoon, he did not eat 50 pieces of pizza in the morning.

(c) If Zhubeen feels very sick in the afternoon, he ate 50 pieces of pizza in the morning.

(2) (Warblers) Suppose the following three statements are true:

(S1) Every warbler has either a blue stripe or a red stripe (but not both).

(S2) If a warbler has a red stripe, it sings beautiful songs all day long.

(S3) If a warbler is sad, it does not sing beautiful songs all day long.

Which of the following can we logically conclude?

(a) If a warbler has a blue stripe, it does not sing beautiful songs all day long.

(b) If a warbler is sad, then it has a blue stripe.

(c) If a warbler sings beautiful songs all day long, it has a red stripe.

(d) If a warbler has a blue stripe, then it is sad.

Suppose we know that if statement P is true, then Q and $\neg Q$ are both true–that is, we prove that $P \implies (Q \text{ and } \neg Q)$. It can't be the case that Q and $\neg Q$ are both true, so this means that *if* P were true, then something *false* would also be true. Therefore P cannot be true.

A statement like Q and $\neg Q$ is called a *logical* contradiction, and an argument like the above to show P is false is called a *proof by contradiction*.

Remember rational versus irrational? A rational number is one that can be written as the ratio of integers (whole numbers). 4/7 is rational, -34/5 is rational, 21 is rational, and 9.39 is rational. But...

Theorem: $\sqrt{2}$ is irrational.

Proof: The statement we ultimately want to prove is that $\sqrt{2}$ is irrational, but we will do so by assuming that $\sqrt{2}$ is *rational* and then using that assumption to prove some contradiction.

Assume (for the sake of argument) $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ for some integers a and b. We can assume a and b are relatively prime.

(3) Why can we assume that?

If we square both sides of $\sqrt{2} = \frac{a}{b}$, we get

$$2 = \frac{a^2}{b^2}, \text{ which is the same as}$$

$$2b^2 = a^2.$$

But since the left hand side is even (divisible by 2), the right hand side must be even as well. If a^2 is even, then a is even. (Why?) So we can write $a = 2c$ for some value c.

Then the equation is

$$2b^2 = (2c)^2, \text{ which leads to}$$

$$2b^2 = 4c^2, \text{ and then dividing both sides by } 2 \text{ gives}$$

$$b^2 = 2c^2.$$

But now the right side of the equation is even, which means the left hand side must be even also, and if b^2 is even, then b must be even.

(4) I claim this contradicts something we said above... what is the contradiction?

(5) Prove that if $x + y$ is irrational, then at least one of x or y is irrational. (Hint: Contrapositive.)

(6) John is confused about the proof that $\sqrt{2}$ is irrational. "Why doesn't the same line of reasoning also prove that $\sqrt{4}$ is irrational?" he wonders. Where does the argument break down? (Write down the same argument as above, except replace $\sqrt{2}$ with $\sqrt{4}$ and make the obvious changes. When does something go wrong? Show your assistant.)

(7) Show that there exist two numbers a and b, both irrational (possibly the same number), such that a^b (a raised to the bth power) is rational.

(Important Hint: I know $\sqrt{2}$ is irrational, so if $\sqrt{2}^{\sqrt{2}}$ is rational, then the statement is true. What if $\sqrt{2}^{\sqrt{2}}$ is irrational, though?

(8) Sam is very pleased with his most recent purchase, a shiny book called the *Complete Non-Self-Reference Reference (CNSRR)*. The ads tell us that this book mentions exactly all the books that doesn't mention themselves. For instance, *Harry Potter and the Sorcerer's Stone* doesn't mention itself, as a book, anywhere in the text, so *CNSRR* mentions it. On the other hand, *America (The Book)* does mention itself, so it does not get mentioned in *CNSRR*.

When Edanel hears about Sam's new purchase, he gets worried. "I think there's a problem with your new book," he says. What does Edanel mean? What's the problem?

(9) In a certain small town in Idaho, there is a curious law: "The town barber shaves exactly those men who do not shave themselves." Do you see any problem with this law?

Theorem: There are infinitely many prime numbers.

Proof: By contradiction. **Assume** that the theorem is false, and that there is only a finite list of prime numbers, say 2, 3, 5, 7, 11, $\ldots$, p_K.

Consider the number we get by multiplying them all together and adding 1:

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot p_K + 1$$

N is not divisible by any of the primes in our list, since it is one more than a multiple of each of them. But N is definitely a number, and every number is divisible by some prime number. This is a contradiction.

Therefore our original **assumption** must have been *FALSE*... Since our assumption was that there are finitely many primes, it follows that in fact there are infinitely many primes.

(10) Suppose I start with a finite set of prime numbers $p_1, p_2, p_3, \ldots, p_K$. If I multiply them together and add 1 to get the number N, as in the proof of infinitely many primes, is it true that N must be a prime number? Why or why not?

(11) There are n people at a party, and some of them have shook hands with each other. Prove that there is some pair of people that have participated in the same number of handshakes. (You may assume that no pair of people shakes hands more than once.)

(Hint: What are the different possible numbers of handshakes each person could have made?)

(12) We saw the first De Morgan's Law, namely that $\neg(P \text{ and } Q) \iff \neg P \text{ or } \neg Q$.
Use truth tables to prove the second De Morgan's Law:
$\neg(P \text{ or } Q) \iff \neg P \text{ and } \neg Q$.

(13) Write a truth table for the following expression:
$((P \implies Q) \implies P) \implies P$

(14) How many different logical operators are there on two logical variables? Explain your answer.

(15) (Paint Buckets) I have several buckets of magical paint (including colors such as *Green-Onion Red*, *Existentialist Yellow*, and *Pajama-Pants Blue*). I can take any two paint colors, mix them together, and get a new paint color (which might actually be the same as one of the original colors). No matter how I pick two colors and mix them together, I always get a color that I already owned.

There is a very special color called ***Crystal Clear*** which has the property that no matter what color I mix it with, I always get that color back as the result. (So *Crystal Clear* + *Existentialist Yellow* = *Existentialist Yellow*, and *Pajama-Pants Blue* + *Crystal Clear* = *Pajama-Pants Blue*, and *Crystal Clear* + *Crystal Clear* = *Crystal Clear*).

Every color X has at least one other color Y which is an **anti-color** for X. If I mix two anti-colors together, I get *Crystal Clear* as the resulting color.

Oh, and one more thing I should mention. One of the magical properties of this paint is that it's possible to mix a color with itself and get a *different* color as the result (weird, right?). Nevertheless, color mixes don't change: If *Green-Onion Red* + *Cow Purple* = *Magic Magenta*, then it will forever and always be the case that *Green-Onion Red* + *Cow Purple* = *Magic Magenta*. This holds true for mixing colors in different amounts, and different orders, also. For instance if I mix *Red* with *Green*, and separately mix *Green* with *Purple*, and then mix the two resulting colors together, the final result is the same as if I mixe *Red* with *Purple*, and separately mix *Green* with *Green*, and then mix those results together. (Because in both cases I ultimately mixed 1 part *Red*, 1 part *Purple*, and 2 parts *Green*).

(a) My friend tells me that he took a bucket of my paint labeled *Shmorange Orange* and mixed it with paint from a second bucket whose label was worn off. The resulting color was still Shmorange Orange. Your friend says he concluded that the second bucket must have been *Crystal Clear* paint. Is he correct? (Note: It certainly **could have been** *Crystal Clear* paint. The question is, is it **possible** there is another color that mixes with *Shmorange Orange* and gives back *Shmorange Orange* as the result?) (Hint: What happens if you mix in an anti-color for Shmorange Orange?)

(b) We know that each color has *at least one* anti-color. Prove that each color has *exactly one* anti-color. (Hint: Start with an arbitrary color X. What if Y is an anti-color, and Z is also an anti-color?)

(c) Suppose I had exactly three colors–*Crystal Clear*, *Existentialist Yellow*, and *Pajama-Pants Blue*. (Clear, Yellow, and Blue for short, or C, Y, and B for really short). Figure out what the following mixes give you:

Clear + *Clear* =

Clear + *Yellow* =

Clear + *Blue* =

Yellow + *Yellow* =

Yellow + *Blue* =

Blue + *Blue* =