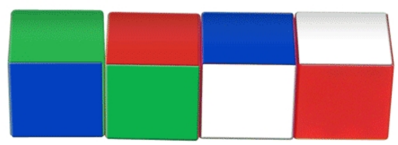
Graph Theory and Instant Insanity

Prepared by Mark on August 19, 2022
Based on a handout by Oleg Gleizer

Part 1: Instant Insanity

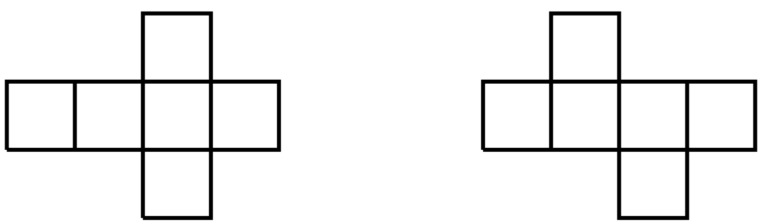
The puzzle you have in front of you is called *Instant Insanity.*
It consists of four cubes, with faces colored with four colors: red, blue, green, and white. The objective is to put the cubes in a row so that each side, front, back, upper, and lower, of the row shows each of the four colors.



There are 41,472 different arrangements of the cubes. Only one is a solution. Finding it by trial and error is quite difficult, but we have witnessed a few students do just that.
However, that rarely happens. We'd like to solve this puzzle today, and to do that, we'll need a few tools.

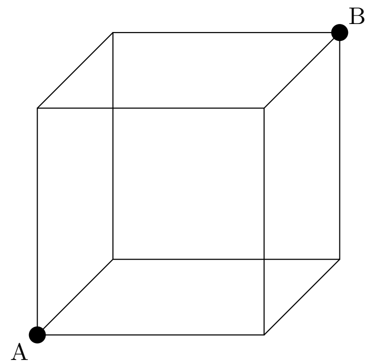
Part 2: Cubic Nets

A *cubic net* is a 2D picture simultaneously showing all the six sides (a.k.a. faces) of a 3D cube, please take a look at the examples below.

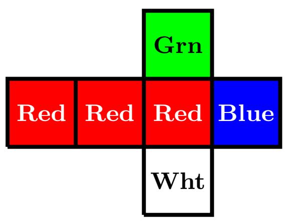


Problem 1:
Draw a cubic net different from the two above.

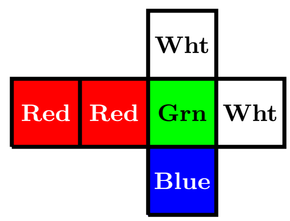
Problem 2:
An ant wants to crawl from point A of a cubic room to the opposite point B, as in the picture below.



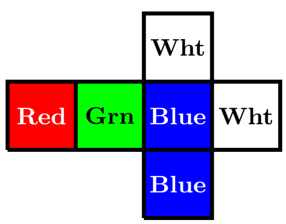
The insect can crawl on any surface, a floor, ceiling, or wall, but cannot fly through the air. Find at least two different shortest paths for the ant (there is more than one).
Let's look at the nets of the puzzle's cubes.



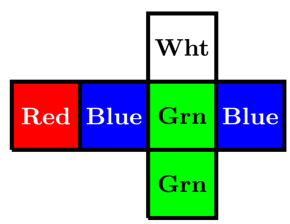
Cube 1



Cube 2



Cube 3



Cube 4

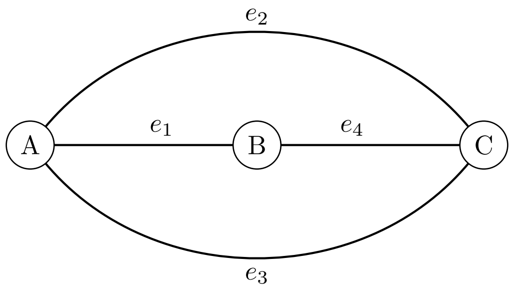
Note that each cube is different.

Part 3: Graphs

Let's remember last week's handout.

Last week's lesson

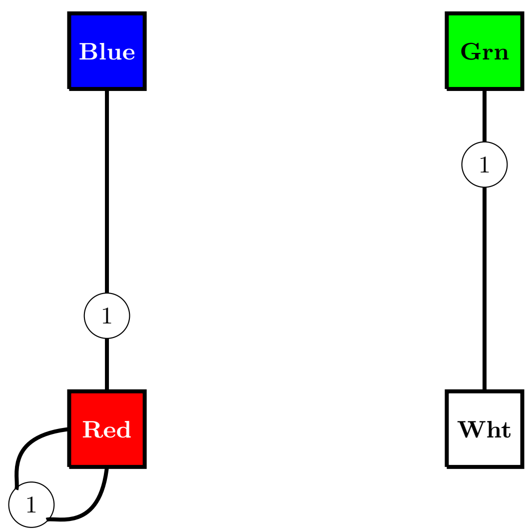
A *graph* is a collection of nodes (vertices) and connections between them (edges). If an edge e connects the vertices v_i and v_j, then we write $e = v_i, v_j$. An example is below.



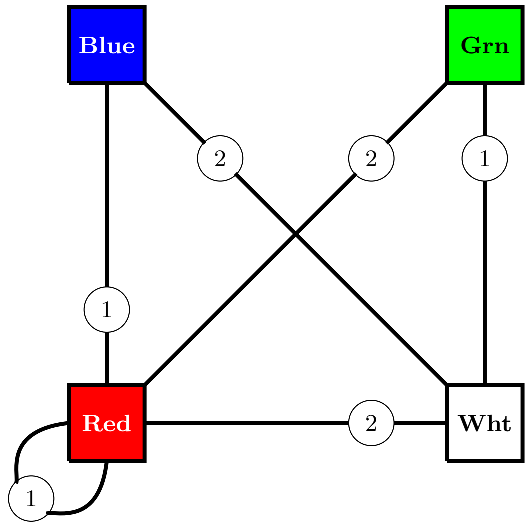
More formally, a graph is defined by a set of vertices $\{v_1, v_2, ...\}$, and a set of edges $\{\ \{v_1, v_2\}, \{v_1, v_3\}, ...\ \}$.

If the order of the vertices in an edge does not matter, a graph is called *undirected.* A graph is called a *directed graph* if the order of the vertices does matter. For example, the (undirected) graph above has three vertices, A, B, and C, and four edges, $e_1 = \{A, B\}$, $e_2 = \{A, C\}$, $e_3 = \{A, C\}$, and $e_4 = \{B, C\}$.

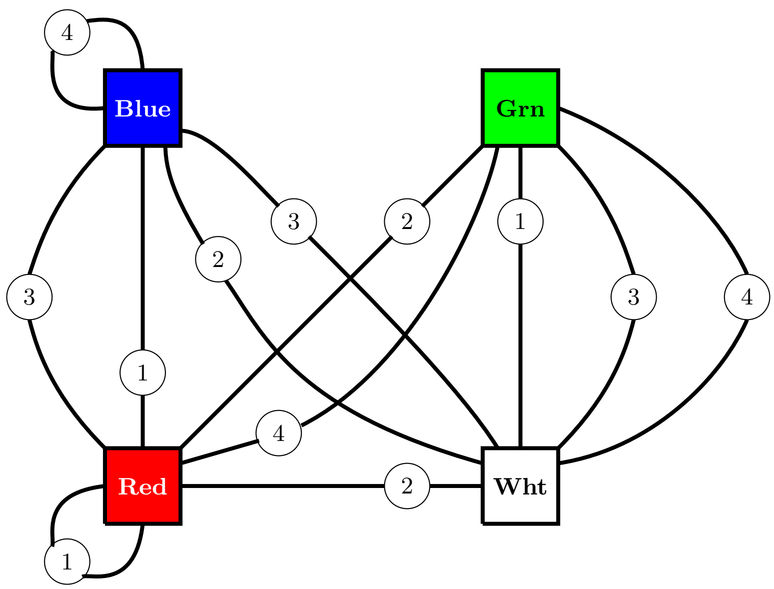
Let us represent Cube 1 by a graph.
The vertices will be the face colors: Blue, Green, Red, and White, so $V = \{B, G, R, W\}$.
Two vertices are be connected by an edge if and only if the corresponding faces are opposing each other on the cube.
Cube 1 has the following edges: $e_1 = \{B, R\}$, $e_2 = \{G, W\}$, and the loop $e_3 = \{R, R\}$. To emphasize that all the three edges represent the first cube, let us mark them with the number 1.



Cube 2 has the following pairs of opposing faces, $\{B, W\}$, $\{G, R\}$, and $\{R, W\}$. Let us add them to the graph as the edges e_4, e_5, and e_6.

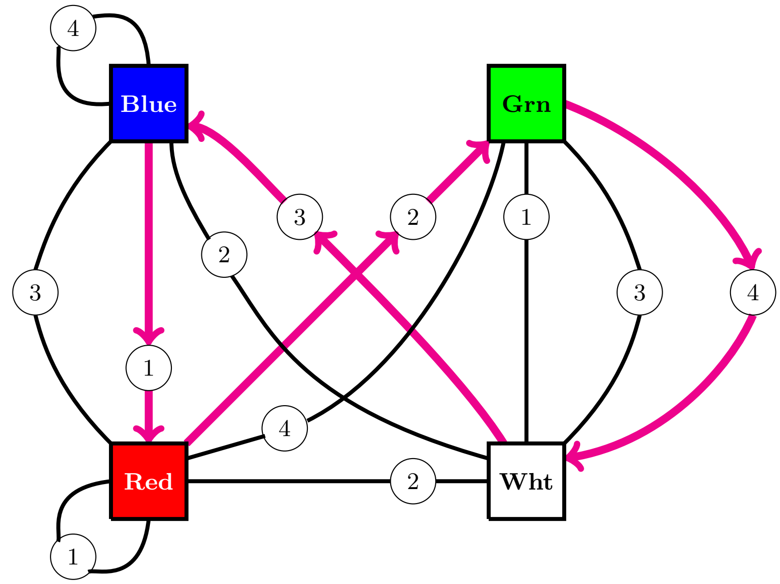


Let us now make the graph representing all four cubes.

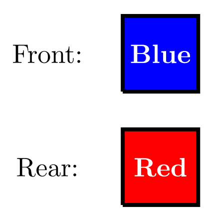


Problem 3:
Check if the above representation is correct for Cubes 3 and 4.

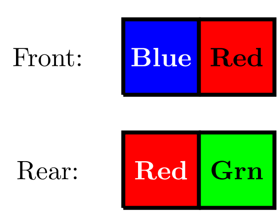
With the help of the above graph, solving the puzzle becomes as easy as a walk in the park, literally. Imagine that the vertices of the above graph are the clearings and the edges are the paths. An edge marked by the number i represents two opposing faces of the i-th cube. Let us try to find a closed walk, a.k.a. a cycle, in the graph that visits each clearing once and uses the paths marked by the different numbers, $i = 1, 2, 3, 4$. If we order the front and rear sides of the cubes accordingly, then the front and rear of the stack will show all the four different colors in the order prescribed by our walk. For example, here is such an (oriented) cycle, represented by the magenta arrows on the picture below.



The first leg of the walk tells us to take Cube 1 and to make sure that its blue side is facing forward. Then the red side, opposite to the blue one, will face the rear.



The next leg of the walk tells us to take Cube 2 and to place it in such a way that its red side faces us while the opposing green side faces the rear. Since we go in a cycle that visits all the colors one-by-one, neither color repeats the ones already used on their sides of the stack.



The third leg of the walk tells us to take Cube 4, not Cube 3, and to place it green side forward, white side facing the rear.

Front:	**Blue**	**Red**		**Grn**
Rear:	**Red**	**Grn**		**Wht**

Finally, the last leg of the walk tells us to take Cube 3 and to place it the white side facing forward, the opposite blue side facing the rear.

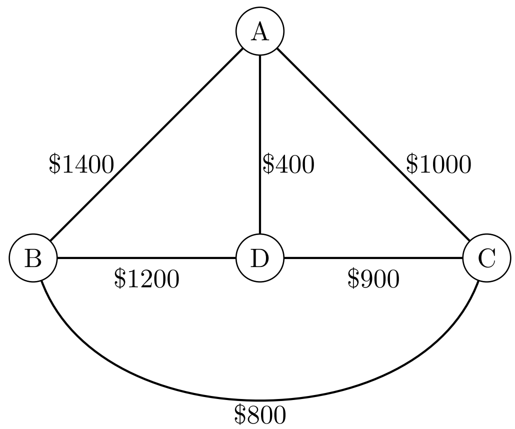
Front:	**Blue**	**Red**	**Wht**	**Grn**
Rear:	**Red**	**Grn**	**Blue**	**Wht**

Now the front and rear of the stack are done. If we manage to find a second oriented cycle in the original graph that has all the properties of the first cycle, but uses none of its edges, we would be able to do the upper and lower sides of the stack and to complete the puzzle. Using the edges we have already traversed during our first walk will mess up the front-rear configuration, but there are still a plenty of the edges left!

Problem 4:
Complete the puzzle.

Part 4: Traveling salesman problem

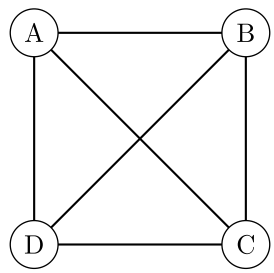
Problem 5:
A salesman with the home office in Albuquerque has to fly to Boston, Chicago, and Denver, visiting each city once, and then to come back to the home office. The airfare prices, shown on the graph below, do not depend on the direction of the travel. Find the cheapest way.
This was on last week's handout, but not everyone had the chance to solve it.



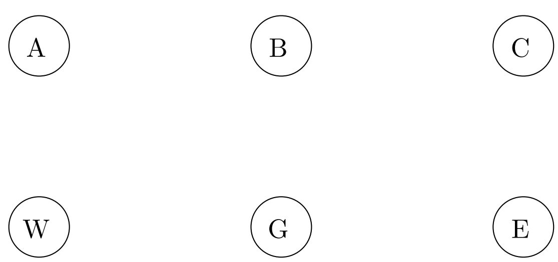
Part 5: Planar graphs

A graph is called *planar*, if it can be drawn in the plane in such a way that no edges cross one another.

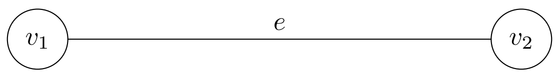
Problem 6:
Show that the following graph is planar.

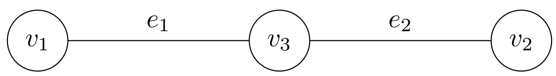


Problem 7:
Is it possible to connect three houses, A, B, and C, to three utility sources, water (W), gas (G), and electricity (E), without using the third dimension, either on the plane or sphere, so that the utility lines do not intersect?



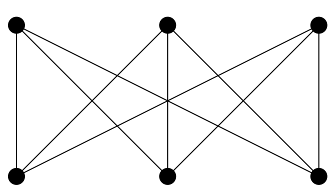
A *subdivision* of a graph G is a graph resulting from the subdivision of the edges of G. The subdivision of an edge $e = (v_1, v_2)$ is a graph containing one new vertex v_3, with the edges $e_1 = (v_1, v_3)$ and $e_2 = (v_3, v_2)$ replacing the edge e.



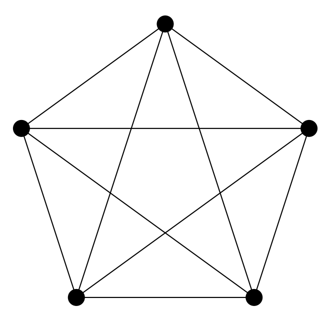


Problem 8:
What is the degree of a subdivision vertex?

A graph H is called a *subgraph* of a graph G if the sets of vertices and edges of H are subsets of the sets of vertices and edges of G.
The following graphs are known as $K_{3,3}$ and K_5.



$K_{3,3}$

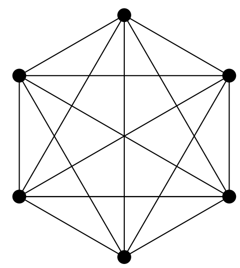


K_5

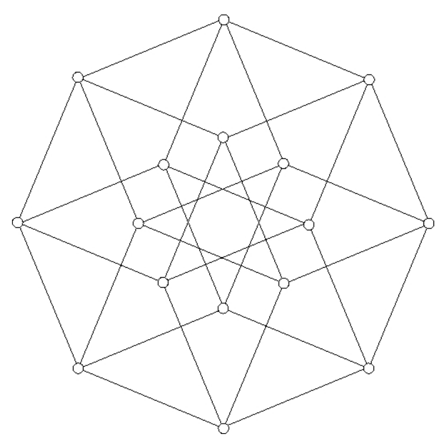
Let H be a graph that is a subdivision of either $K_{3,3}$ or K_5. If H is a subgraph of a graph G, then H is called a *Kuratowski subgraph*, after a famous Polish mathematician Kazimierz Kuratowski (1896-1980).

Theorem 1:
A graph is planar if and only if it has no Kuratowski subgraph.

Problem 9:
Is the following graph planar? Why or why not?

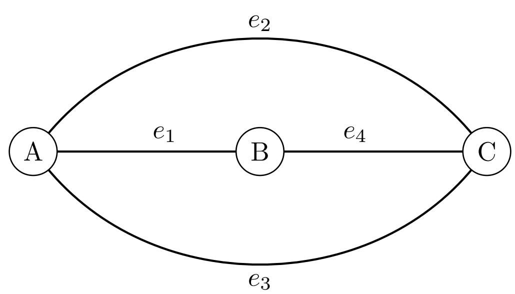


Problem 10:
Is the following graph planar? Why or why not?



Part 6: Euler characteristic

Let G be a planar graph, drawn with no edge intersections. The edges of G divide the plane into regions, called *faces*. The regions enclosed by the graph are called the *interior faces*. The region surrounding the graph is called the *exterior (or infinite) face*. The faces of G include both the interior faces and the exterior one. For example, the following graph has two interior faces, F_1, bounded by the edges e_1, e_2, e_4; and F_2, bounded by the edges e_1, e_3, e_4. Its exterior face, F_3, is bounded by the edges e_2, e_3.

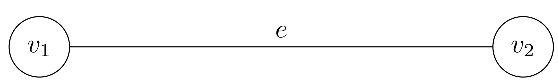


The *Euler characteristic* of a graph is the number of the graph's vertices minus the number of the edges plus the number of the faces.

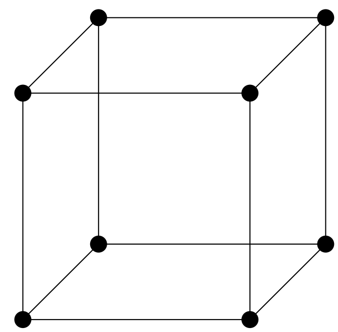
$$\chi = V - E + F \tag{1}$$

Problem 11:
Compute the Euler characteristic of the graph above.

Problem 12:
Compute the Euler characteristic of the following graph.



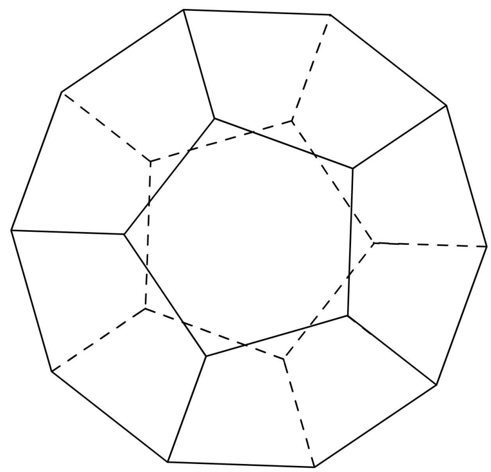
Problem 13:
Is the following graph planar? If you think it is, please re-draw the graph so that it has no intersecting edges. If you think the graph is not planar, please explain why.



Problem 14:
Compute the Euler characteristic of the graph from Problem Problem 13.

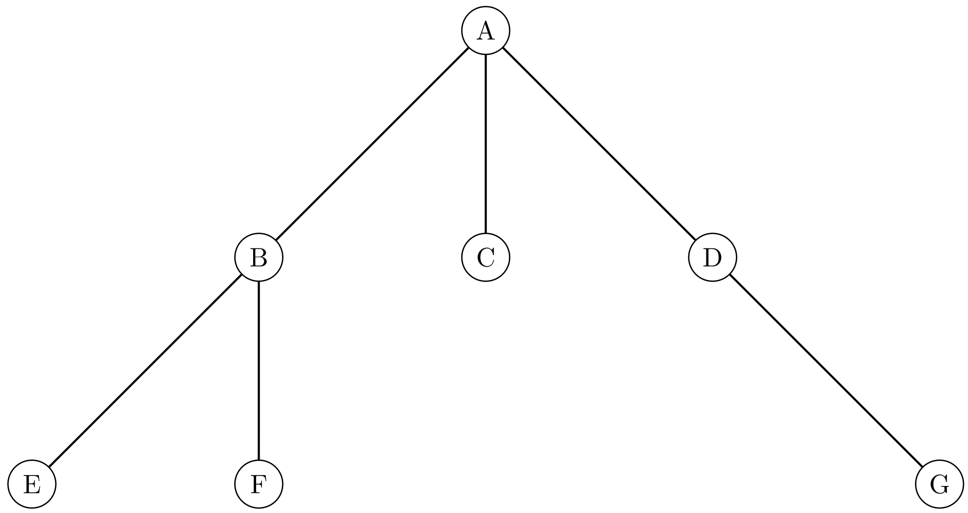
Let us consider the below picture of a *regular dodecahedron* as a graph, the vertices representing those of the graph, and the edges, both solid and dashed, representing the edges of the graph.

Problem 15:
Is the graph planar? If you think it is planar, please re-draw the graph so that it has no intersecting edges. If you think the graph is not planar, please explain why.



Problem 16:
Compute the Euler characteristic of the graph from Problem Problem 15. Can you conjecture what the Euler characteristic of every planar graph is equal to?

A graph is called a *tree* if it is connected and has no cycles. Here is an example.



A path is called *simple* if it does not include any of its edges more than once.

Problem 17:
Prove that a graph in which any two vertices are connected by one and only one simple path is a tree.

Problem 18:
What is the Euler characteristic of a finite tree?

Theorem 2:
Let a finite connected planar graph have V vertices, E edges, and F faces. Then $V - E + F = 2$.

Problem 19:
Prove Theorem Theorem 2. Hint: removing an edge from a cycle does not change the number of vertices and reduces the number of edges and faces by one.

Problem 20:
There are three ponds in a botanical garden, connected by ten non-intersecting brooks so that the ducks can sweem from any pond to any other. How many islands are there in the garden?

Problem 21:
All the vertices of a finite graph have degree three. Prove that the graph has a cycle.

Problem 22:
Draw an infinite tree with every vertex of degree three.

Problem 23:
Prove that a connected finite graph is a tree if and only if $V = E + 1$.

Problem 24:
Give an example of a finite graph that is not a tree, but satisfies the relation $V = E + 1$.