Week 6: Catchup

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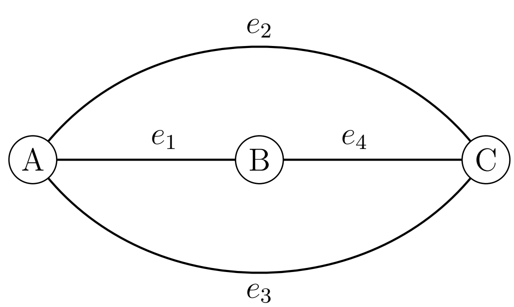
July 31, 2022

1 Euler characteristic

Let G be a planar graph, drawn with no edge intersections. The edges of G divide the plane into regions, called *faces*. The regions enclosed by the graph are called the *interior faces*. The region surrounding the graph is called the *exterior (or infinite) face*. The faces of G include both the interior faces and the exterior one. For example, the following graph has two interior faces, F_1, bounded by the edges e_1, e_2, e_4; and F_2, bounded by the edges e_1, e_3, e_4. Its exterior face, F_3, is bounded by the edges e_2, e_3.

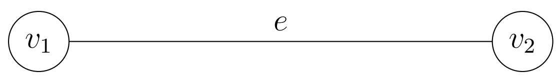


The *Euler characteristic* of a graph is the number of the graph's vertices minus the number of the edges plus the number of the faces.

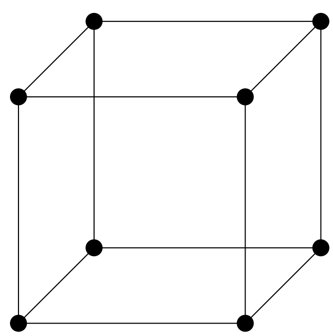
$$\chi = V - E + F \tag{1}$$

Problem 1.1. *Compute the Euler characteristic of the graph above.*

Problem 1.2. *Compute the Euler characteristic of the following graph.*



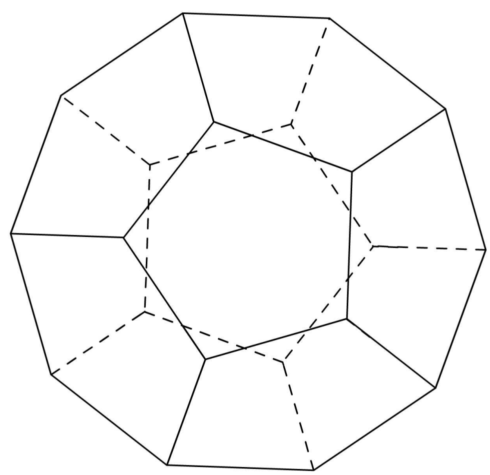
Problem 1.3. *Is the following graph planar? If you think it is, please redraw the graph so that it has no intersecting edges. If you think the graph is not planar, please explain why.*



Problem 1.4. *Compute the Euler characteristic of the graph from Problem 1.3.*

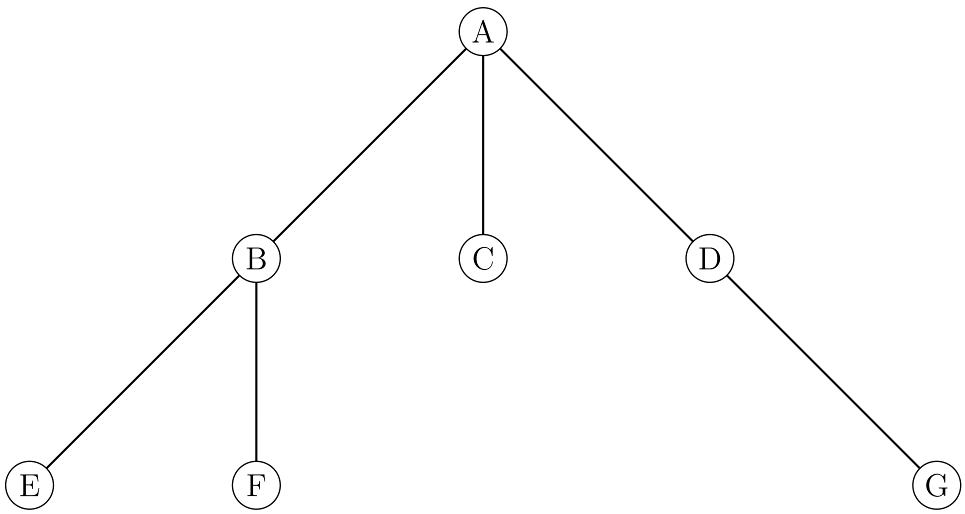
Let us consider the below picture of a *regular dodecahedron* as a graph, the vertices of the polytope representing those of the graph, the edges of the polytope, both solid and dashed, representing the edges of the graph.

Problem 1.5. *Is the graph planar? If you think it is planar, please re-draw the graph so that it has no intersecting edges. If you think the graph is not planar, please explain why.*



Problem 1.6. *Compute the Euler characteristic of the graph from Problem 1.5. Can you conjecture what the Euler characteristic of every planar graph is equal to?*

A graph is called a *tree* if it is connected and has no cycles. Here is an example.



A path is called *simple* if it does not include any of its edges more than once.

Problem 1.7. *Prove that a graph in which any two vertices are connected by one and only one simple path is a tree.*

Problem 1.8. *What is the Euler characteristic of a finite tree?*

Theorem 1. *Let a finite connected planar graph have V vertices, E edges, and F faces. Then $V - E + F = 2$.*

Problem 1.9. *Prove Theorem 1. Hint: removing an edge from a cycle does not change the number of vertices and reduces the number of edges and faces by one.*

Problem 1.10. *There are three ponds in a botanical garden, connected by ten non-intersecting brooks so that the ducks can swim from any pond to any other. How many islands are there in the garden?*

Problem 1.11. *All the vertices of a finite graph have degree three. Prove that the graph has a cycle.*

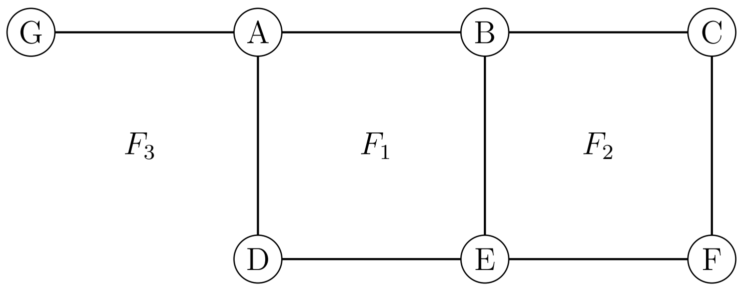
Problem 1.12. *Draw an infinite tree with every vertex of degree three.*

Problem 1.13. *Prove that a connected finite graph is a tree if and only if* $V = E + 1$.

Problem 1.14. *Give an example of a finite graph that is not a tree, but satisfies the relation* $V = E + 1$.

2 Proving that $K_{3,3}$ and K_5 are not planar

Let G be a planar graph with E edges. Let us call the *degree of its face*, $deg(F_i)$, the number of the edges one needs to traverse to get around the face F_i. For example, the following are the degrees of the faces of the graph below: $deg(F_1) = deg(F_2) = 4$, $deg(F_3) = 8$.



Note that in order to get around the exterior face of the graph, F_3, one has to traverse the edge $\{A, G\}$ twice.

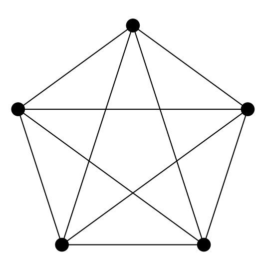
Problem 2.1. *Prove that* $\sum deg(F_i) = 2E$.

A graph is called *simple* if it is undirected, has no loops, and no multiple edges. The latter means that every pair of vertices connected by an edge is connected by only one edge. For example, the graph at the top of this page is simple, the graph at the top of page 1 is not.

Problem 2.2. *Let a finite connected simple planar graph have $E > 1$ edges and F faces. Prove that then $2E \geq 3F$.*

Problem 2.3. *Prove that for a finite connected simple planar graph, $E \leq 3V - 6$.*

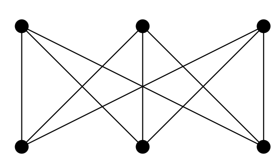
Problem 2.4. *Prove that the graph K_5 is not planar.*



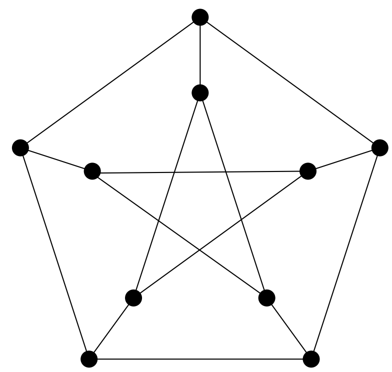
Problem 2.5. *Let G be a finite connected simple planar graph with $E > 1$ edges and no triangular faces. Prove that then $E \geq 2F$.*

Problem 2.6. *Let G be a finite connected simple planar graph with $E > 1$ edges and no triangular faces. Prove that then $E \leq 2V - 4$.*

Problem 2.7. *Prove that the graph $K_{3,3}$ is not planar.*



Problem 2.8. *The following graph is known as the Petersen graph. Is it planar? Why or why not?*



3 Prison Games

In a strange prison, prisoners are often given the opportunity to play a game to go free early. The games are very different, but share one important rule: prisoners may not communicate once the game starts. However, they are given an opportunity to strategize once they hear the rules.

Problem 3.1. *Two prisoners are each given a coin, and are sent to separate rooms. They must each flip their coin, and attempt to guess the other prisoner's coin. If at least one of them guesses right, they both win, and are set free. If they both guess wrong, they both lose, and must remain in prison for life. How can the prisoners guarantee a win?*

Problem 3.2. *Three prisoners are each given a hat. The hats come in 3 colors (black, white and red) and it is possible for multiple prisoners to have the same colored hat. Each prisoner can only see the hats of the other two prisoners, and has to guess the color of their own. If at least one of them guesses right, all of them win and are set free. If all three guess wrong, they all lose and must remain in prison for life. Find a strategy for the prisoners to guarantee their freedom. Can they still find a strategy that always works if two of them need to guess right for all of them to win?*

Problem 3.3. *A single prisoner is blindfolded, taken into a dark room, and brought to a table with 100 coins on it. They are told that 14 coins are heads, and the rest are tails. The prisoner must then split the coins into two groups such that each group has an equal number of heads in it. They can flip coins over, but they can't at any point look at the coins. How can the prisoner accomplish this task? (This isn't a trick question.)*

Problem 3.4. *20 prisoners are kept in isolated cells. Every day, the warden chooses a completely random prisoner and takes them to a room with two switches, where the prisoner has to flip exactly one switch and leave the room. The switches do not connect to anything, and both start in the OFF position. The warden does not guarantee anything about the choice of prisoner except that eventually, all of the prisoners will have visited the room as many times as they can count. At any time, any prisoner can declare that all of the prisoners have visited the switch room. If they are right, the prisoners win and all of them are set free. If they are wrong, the prisoners lose and remain imprisoned for life. How can the prisoners guarantee a win? (Bonus problem: what if the prisoners don't know the initial state of the switches?)*

Challenge Problem 3.5. *This game uses two prisoners. Before the game begins, the warden places a coin on each square of a chessboard. Each coin is randomly flipped either heads or tails. The warden hides the key to the prison under one coin (it's a very small key). The first prisoner is brought to the chessboard, and is shown where the key is. They may then flip one, and only one, coin from heads to tails, or from tails to heads. The first prisoner is then sent away, and the warden rotates and shifts all the coins slightly (so no information can be communicated via the positions/rotations of the coins). The second prisoner is brought to the chessboard, and can lift one, and only one, coin. If the key is under that coin, the prisoners win, and can escape. Otherwise, they lose. How can the prisoners guarantee a win?*

Challenge Problem 3.6. *100 prisoners will play this game. 100 boxes numbered from 1 to 100 are placed in a room, and 100 slips of paper numbered from 1 to 100 are randomly placed in the box, with 1 paper per box. Each prisoner is also assigned a unique number from 1 to 100. One by one, each prisoner is brought into the room, and may open up to 50 boxes. If the prisoner finds the paper with their own number, they have succeeded. Otherwise, they have failed. The prisoner then leaves the room, all the boxes are closed again, and a new prisoner is brought into the room. If every prisoner succeeds, they all win. However, if even one prisoner fails, they all lose. What is a good strategy the prisoners can employ, if they are not allowed to communicate once the game starts. (Note: Unlike the other games, the prisoners can't guarantee a win. However, they can significantly improve their chances from a random guess).*