NUMBER BASES

BEGINNERS 05/03/2015

The numeral system we use in everyday life is base ten. This means that we use ten digits:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

and that the value of a digit depends on its position in a number. For example,

$$2015 = 2 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 5 \cdot 10^0$$

However, numbers can be written in any number base $n \geq 2$.

- Base n uses digits $0, 1, 2, ..., n - 1$.
- If $n > 10$, letters are used instead of digits starting with the digit representing 10.

(1) Fill the table below.

Number/Base	2	3	4	7	8	16
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						
14						
15						
16						

(2) Consider the table on the previous page.
(a) Write n in base n for any n:

(b) Write n^2 in base n for any n:

(c) Write n^{2015} in base n for any n:

(3) Fill out the table below.

Number/Base	2	3	4	8	9	16
22						
38						
54						
100						
232						
1000						
2015						

There are two ways to go to any base from base 10:

(A) The Greedy Algorithm: determines notation from left to right.

- Find the highest power of the base that fits into the given number (can fit more than once!)
- Subtract this power from the number. You can subtract this power as many times as it fits.
- Repeat with the new number.

Example: What is 197_{10} in base 9?
$\mathbf{2} \cdot 9^2 = 162$, $197 - 162 = 35$, $35 < 9^2$, 2 is the left-most digit
$\mathbf{3} \cdot 9^1 = 27$, $35 - 27 = 8$, $8 < 9^1$, 3 is the next digit
$\mathbf{8} \cdot 9^0 = 8$, 8 is the right-most digit
$197_{10} = 238_9$

(B) The Division Algorithm: determines notation from right to left.

- Divide the number by the base and find the remainder.
- Divide the quotient by the base and find the remainder.
- Repeat until you can no longer divide the quotient by the base.

Example: What is 197_{10} in base 9?
$197 : 9 = 21$ Remainder **8**, 8 is the right-most digit
$21 : 9 = 2$ Remainder **3**, 3 is the next digit
$2 : 9 = 0$ Remainder **2**, 2 is the left-most digit
$197_{10} = 238_9$

(1) Solve the following problems using the greedy algorithm:
   (a) What is 27_{10} in base 2?

   (b) What is 121_{10} in base 3?

   (c) What is 284_{10} in base 8?

   (d) What is 299_{10} in base 12?

   (e) What is 342_{10} in base 7?

(2) Solve the following problems using the division algorithm:

(a) What is 57_{10} in base 4?

(b) What is 275_{10} in base 7?

(c) What is 143_{10} in base 8?

(d) What is 342_{10} in base 9?

(e) What is 432_{10} in base 15?

(3) Which method do you prefer? Explain.

When converting numbers with base greater than 10, we must remember that letters are used instead of digits.

(A) Consider the conversion between base 16 and base 10.

Example: What is $1FDA_{16}$ in base 10?
$1FDA_{16} = \mathbf{1} \cdot 16^3 + \mathbf{15} \cdot 16^2 + \mathbf{13} \cdot 16^1 + \mathbf{10} \cdot 16^0$
$1FDA_{16} = 1 \cdot 4096 + 15 \cdot 256 + 13 \cdot 16 + 10$
$1FDA_{16} = 8154_{10}$

(1) Computer the following powers of 16.
(a) $16^1 =$

(b) $16^2 =$

(c) $16^3 =$

(d) $16^4 =$

(2) Convert the following numbers between base 16 and base 10:
(a) What is 1984_{10} in base 16?

(b) What is 4813_{10} in base 16?

(c) What is 1984_{16} in base 10?

(d) What is $17EC_{16}$ in base 10?

Sometimes, we can also convert numbers with different bases without looking at base 10 intermediates. This works very well when the bases are powers of each other.

(B) Let us start with the conversion between base 2 and base 4.

Since $4 = 2^2$, we can convert consecutive pairs of digits in base 2 to single digits in base 4. Conversely, we can convert each digit in base 4 to a pair of digits in base 2.

Example 1: What is 12_4 in base 2?
$12_4 = 1 \cdot 4^1 + 2 \cdot 4^0 = 1 \cdot 4 + 2 \cdot 1 = \mathbf{1} \cdot 2^2 + \mathbf{1} \cdot 2^1 + \mathbf{0} \cdot 2^0 = 110_2$
Another way to do this is to look at the digits individually,
$1_4 = 1_2$ and $2_4 = 10_2$
Thus, $12_4 = 110_2$

Example 2: What is 11011001_2 in base 4?
To convert the given number, we separate the number in pairs of digits starting from the right. If the number has an odd number of digits, we can place a 0 in the beginning. We can then convert each pair of digits to base 4. For instance,
$100_2 = [01]\,[00]_2 = [1]\,[0]_4 = 10_4$
$1000_2 = [10]\,[00]_2 = [2]\,[0]_4 = 20_4$
Thus, $11011001_2 = [11]\,[01]\,[10]\,[01]_2 = [3]\,[1]\,[2]\,[1]_4 = 3121_4$

(1) Explain why $1_2 = 1_4$.

(2) Explain why $10000_2 = 100_4$.

(3) Convert the following numbers between base 2 and base 4:
  (a) What is 2222_4 in base 2?

  (b) What is 11232_4 in base 2?

(c) What is 110101_2 in base 4?

(d) What is 1011011_2 in base 4?

(C) Next let us consider the conversion between base 2 and base 8.

Since $8 = 2^3$, we can convert consecutive triplets of digits in base to single digits in base 8. Conversely, we can convert each digit in base 8 to three digits in base 2.

Example 1: What is 1001011_2 in base 8?

To convert the given number, we separate the number in sets of three digits starting from the right. If the last set does not have three digits, we can place one or two $0s$ in the beginning as necessary. We can then convert each set of three digits to base 8.

$1001011_2 = [001]\,[001]\,[011]\,_2 = [1]\,[1]\,[3]\,_8 = 113_8$

Example 2: What is 531_8in base 2?

$531_8 = [5]\,[3]\,[1]_8 = [101]\,[011]\,[001]\,_2 = 101011001_2$

(1) Convert the following numbers between base 2 and base 8:

(a) What is 547_8 in base 2?

(b) What is 3700_8 in base 2?

(c) What is 110001011_2 in base 8?

(d) What is 111100111_2 in base 8?

(D) Consider the conversion between base 2 and base 16.

Since $16 = 2^4$, we can convert consecutive sets of four digits in base 2 to single digits in base 16. Conversely, we can convert each digit in base 16 to four digits in base 2.

Example 1: What is 1011100011_2 in base 16?

To convert the given number, we separate the number in sets of four digits starting from the right. If the last set does not have four digits, we can place one, two or three $0s$ in the beginning as necessary. We can then convert each set of four digits to base 16.

$1011100011_2 = [0010]\,[1110]\,[0011]_2 = [2]\,[14]\,[3]_{16} = [2]\,[E]\,[3]_{16} = 2E3_{16}$

Example 2: What is $EE8_{16}$in base 2?

$EE8_{16} = [1110]\,[1110]\,[1000]\,_2 = 1110111010001_2$

(1) Convert the following numbers between base 2 and base 16:

(a) What is 1234_{16} in base 2?

(b) What is $ABCD_{16}$ in base 2?

(c) What is 1111011101111_2 in base 16?

(d) What is 101111101100111_2 in base 16?

(2) Convert the following numbers between base 3 and base 9:
   (a) What is 348_9 in base 3?

   (b) What is 2028_9 in base 3?

   (c) What is 2102_3 in base 9?

   (d) What is 122110021_3 in base 9?

(3) Convert the following numbers between base n and base n^2:
(a) What is 45_{n^2} in base n where $n^2 \geq 6$?

(b) What is $H10_{n^2}$ in base n where $n^2 \geq 17$?

(c) What is 12340_n in base n^2 where $n \geq 5$?

(d) What is $ABCD_n$ in base n^2 where $n \geq 16$?

(E) Consider the conversion between base 8 and base 16 via base 2.

Example: What is FEA_{16} in base 8?
FEA_{16} = $[1111]\,[1110]\,[1010]_2$ =$[111]\,[111]\,[101]\,[010]_2$ = $[7]\,[7]\,[5]\,[2]_8$ = 7752_8

(1) Convert the following numbers between base 8 and base 16:
(a) What is $4AE_{16}$ in base 8?

(b) What is $ACFD1_{16}$ in base 8?

(c) What is 350_8 in base 16?

(d) What is 4176_8 in base 16?