Week 4: Planar Graphs

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Shortest Path Algorithms

A problem related to the Traveling Salesman is the *Shortest Path Problem*, which asks for the path with least weight from vertex a to vertex b. Unlike the Traveling Salesman problem, which has no known fast solution, there are many algorithms for quickly finding the shortest path in a weighted graph. One such algorithm was created by Edsger W. Dijkstra. Dijkstra's algorithm goes as follows:

Dijkstra's Algorithm. Let $L(x)$ be the distance from vertex a to vertex x. The following steps will give us $L(b)$

1. Let $L(a) = 0$, and $L(x) = \infty$ for all other x. These values will be refined as the process goes on. Initialize set $T = \mathcal{V}$, the set of all vertices.

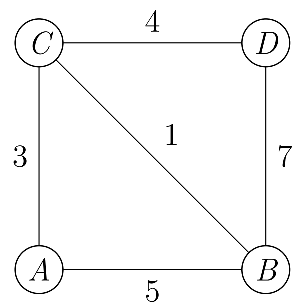
2. While vertex b is in T:

   (a) Set v to be a vertex in T with the minimum $L(v)$.

   (b) Remove v from set T.

   (c) For each vertex x in T adjacent to v, set $L(x)$ to the minimum of $L(x)$ and $L(v) + w(v,x)$, where $w(v,x)$ is the weight of the edge connecting v and x.

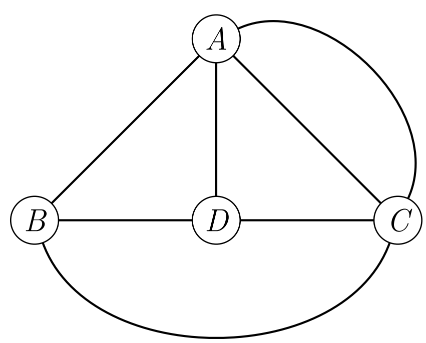
Problem 1. *Use Dijkstra's Algorithm to find the shortest path from vertex A to B in the following graph.*



Minimum Spanning Tree

Another related problem is the Minimum Spanning Tree problem. A *tree* is a graph without any cycles. This means that there is a unique path between any two vertices. Given a graph G with vertices $\mathcal{V}$ and edges $\mathcal{E}$, a *spanning tree* of G is a connected subgraph of G with the same vertex set $\mathcal{V}$, and without any cycles.

Problem 2. *Find a spanning tree of the following graph.*



Given a weighted graph G, the *minimum spanning tree* (MST) of G is a spanning tree of G with minimum total edge weight. There are multiple algorithms for efficiently finding the minimum spanning tree of a weighted graph. One such algorithm is Prim's algorithm, which goes as follows:

Prim's Algorithm. Let $C(x)$ be the cost of adding vertex x to the MST and $E(x)$ be the edge used to add vertex x to the MST. The following steps will give us the MST:

1. Label any vertex v_0 as the first vertex added to the MST. Let $C(v_0) = 0$, and $C(x) = \infty$ for all other vertices x. Let $E(x)$ be empty for all vertices x. Initialize set $Q = \mathcal{V}$, the set of all vertices.

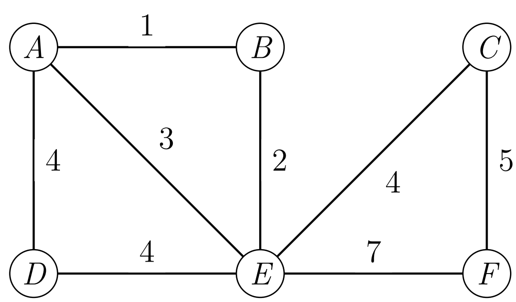
2. While Q is non-empty:

   (a) Remove vertex v from Q with minimum value of $C(v)$.

   (b) Add vertex v and edge $E(v)$ (if $E(v)$ is not empty) to the MST.

   (c) For each vertex x in Q that is adjacent to v, if $w(v, x) < C(x)$, set $C(x) = w(v, x)$ and $E(x) = \{v, x\}$.

Problem 3. *Use Prim's Algorithm to find a minimum spanning tree in the following graph.*



Problem 4. *Is the path between two vertices on an MST always the same as the shortest path on the original graph?*

Challenge Problems

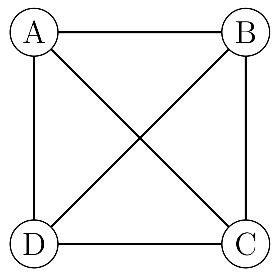
Problem 5. *Prove that Dijkstra's algorithm gives the shortest path between two vertices on a weighted graph.*

Problem 6. *Prove that Prim's algorithm gives the minimum spanning tree on a weighted graph.*

Problem 7. *So far we have only looked at graphs with positively weighted edges. Will Dijkstra's algorithm and Prim's algorithm also work correctly on graphs with negatively weighted edges?*

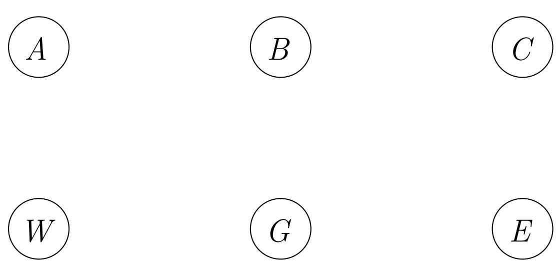
Planar graphs

A graph is called *planar*, if it can be drawn in the plane in such a way that no edges cross one another. For example, the following graph is planar.

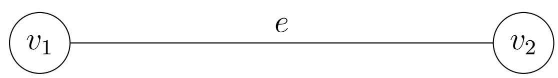


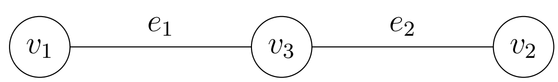
Problem 8. *Demonstrate that the above graph is planar by drawing it in such a way that no edges cross one another.*

Problem 9. *Is it possible to connect three houses, A, B, and C, to three utility sources, water (W), gas (G), and electricity (E), without using the third dimension so that the utility lines do not intersect?*



A *subdivision* of a graph G is a graph resulting from the subdivision of the edges of G. The subdivision of an edge $e = (v_1, v_2)$ is a graph containing one new vertex v_3, with the edges $e_1 = (v_1, v_3)$ and $e_2 = (v_3, v_2)$ replacing the edge e.

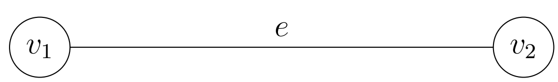




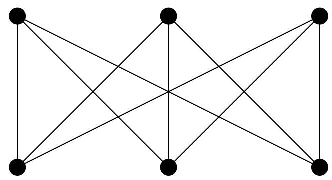
Problem 10. *What is the degree of a subdivision vertex?*

A graph H is called a *subgraph* of a graph G if the sets of vertices and edges of H are subsets of the sets of vertices and edges of G.

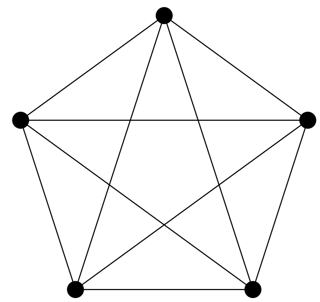
Problem 11. *How many possible subgraphs does the following graph have?*



The following graphs are known as $K_{3,3}$ and K_5.



$K_{3,3}$



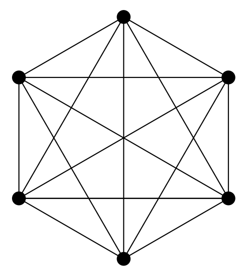
K_5

Let H be a graph that is a subdivision of either $K_{3,3}$ or K_5. If H is a subgraph of a graph G, then H is called a *Kuratowski subgraph*, after a famous Polish mathematician Kazimierz Kuratowski (1896-1980).

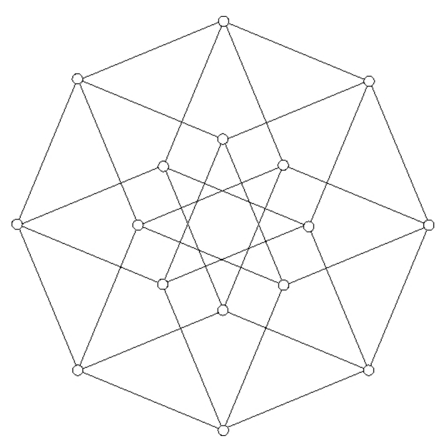
Kazimierz Kuratowski

Theorem 1. *(Kuratowski) A graph is planar if and only if it has no Kuratowski subgraph.*

Problem 12. *Is the following graph planar? Why or why not?*

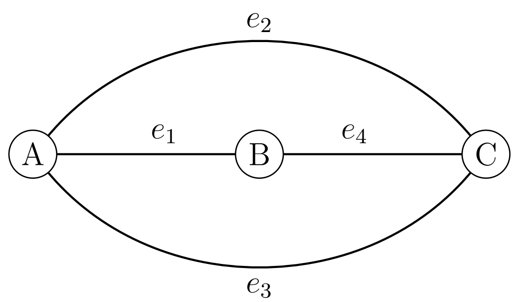


Problem 13. *Is the following graph planar? Why or why not?*



Euler characteristic

Let G be a planar graph, drawn with no edge intersections. The edges of G divide the plane into regions, called *faces*. The regions enclosed by the graph are called the *interior faces*. The region surrounding the graph is called the *exterior (or infinite) face*. The faces of G include both the interior faces and the exterior one. For example, the following graph has two interior faces, F_1, bounded by the edges e_1, e_2, e_4; and F_2, bounded by the edges e_1, e_3, e_4. Its exterior face, F_3, is bounded by the edges e_2, e_3.

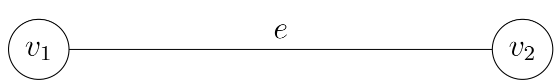


The *Euler characteristic* of a graph is the number of the graph's vertices minus the number of the edges plus the number of the faces.

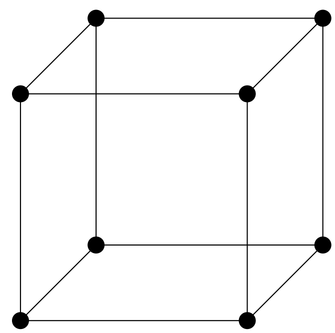
$$\chi = V - E + F \tag{1}$$

Problem 14. *Compute the Euler characteristic of the graph above.*

Problem 15. *Compute the Euler characteristic of the following graph.*



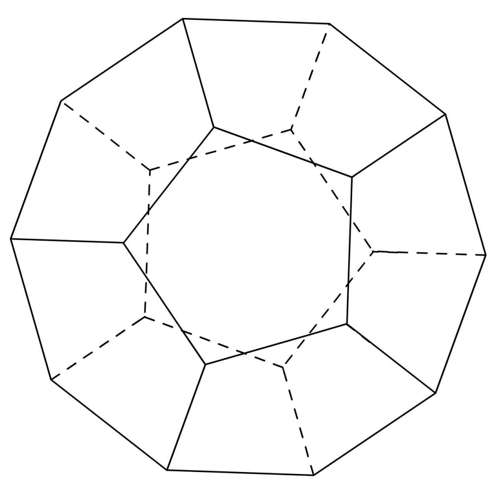
Problem 16. *Is the following graph planar? If you think it is, please re-draw the graph so that it has no intersecting edges. If you think the graph is not planar, please explain why.*



Problem 17. *Compute the Euler characteristic of the graph from Problem 16.*

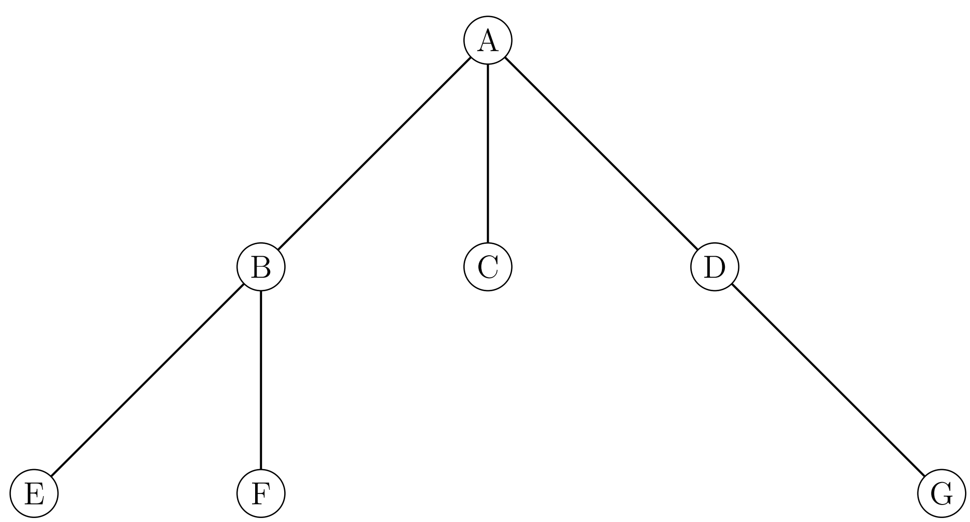
Let us consider the below picture of a *regular dodecahedron* as a graph, the vertices of the polytope representing those of the graph, the edges of the polytope, both solid and dashed, representing the edges of the graph.

Problem 18. *Is the graph planar? If you think it is planar, please re-draw the graph so that it has no intersecting edges. If you think the graph is not planar, please explain why.*



Problem 19. *Compute the Euler characteristic of the graph from Problem 18. Can you conjecture what the Euler characteristic of every planar graph is equal to?*

A graph is called a *tree* if it is connected and has no cycles. Here is an example.



A path is called *simple* if it does not include any of its edges more than once.

Problem 20. *Prove that a graph in which any two vertices are connected by one and only one simple path is a tree.*

Problem 21. *What is the Euler characteristic of a finite tree?*

Theorem 2. *Let a finite connected planar graph have V vertices, E edges, and F faces. Then $V - E + F = 2$.*

Problem 22. *Prove Theorem 2. Hint: removing an edge from a cycle does not change the number of vertices and reduces the number of edges and faces by one.*

Problem 23. *There are three ponds in a botanical garden, connected by ten non-intersecting brooks so that the ducks can swim from any pond to any other. How many islands are there in the garden?*

Problem 24. *All the vertices of a finite graph have degree three. Prove that the graph has a cycle.*

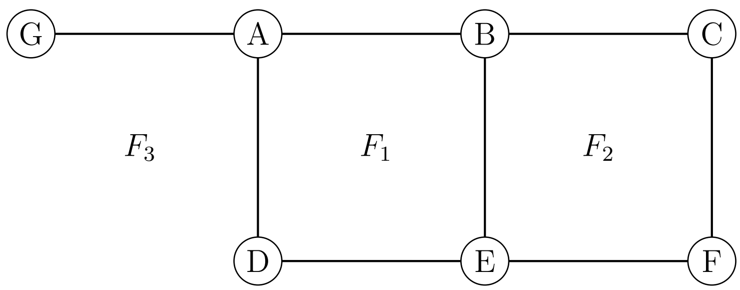
Problem 25. *Draw an infinite tree with every vertex of degree three.*

Problem 26. *Prove that a connected finite graph is a tree if and only if* $V = E + 1$.

Problem 27. *Give an example of a finite graph that is not a tree, but satisfies the relation* $V = E + 1$.

(Challenge) Proving that $K_{3,3}$ and K_5 are not planar

Let G be a planar graph with E edges. Let us call the *degree of its face*, $deg(F_i)$, the number of the edges one needs to traverse to get around the face F_i. For example, the following are the degrees of the faces of the graph below: $deg(F_1) = deg(F_2) = 4$, $deg(F_3) = 8$.



Note that in order to get around the exterior face of the graph, F_3, one has to traverse the edge $\{A, G\}$ twice.

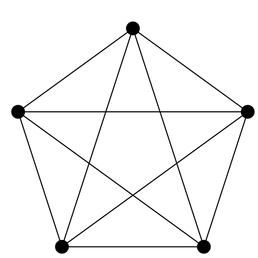
Problem 28. *Prove that* $\sum deg(F_i) = 2E$.

A graph is called *simple* if it is undirected, has no loops, and no multiple edges. The latter means that every pair of vertices connected by an edge is connected by only one edge. For example, the graph at the top of this page is simple, the graph at the top of page 9 is not.

Problem 29. *Let a finite connected simple planar graph have $E > 1$ edges and F faces. Prove that then $2E \geq 3F$.*

Problem 30. *Prove that for a finite connected simple planar graph, $E \leq 3V - 6$.*

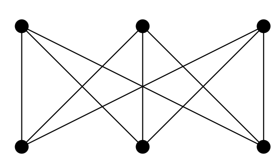
Problem 31. *Prove that the graph K_5 is not planar.*



Problem 32. *Let G be a finite connected simple planar graph with $E > 1$ edges and no triangular faces. Prove that then $E \geq 2F$.*

Problem 33. *Let G be a finite connected simple planar graph with $E > 1$ edges and no triangular faces. Prove that then $E \leq 2V - 4$.*

Problem 34. *Prove that the graph $K_{3,3}$ is not planar.*



Problem 35. *The following graph is known as the Petersen graph. Is it planar? Why or why not?*