Week 2: Properties of Graphs and Eulerian Cycles

August Deer, Siddarth Chalasani, Oleg Gleizer

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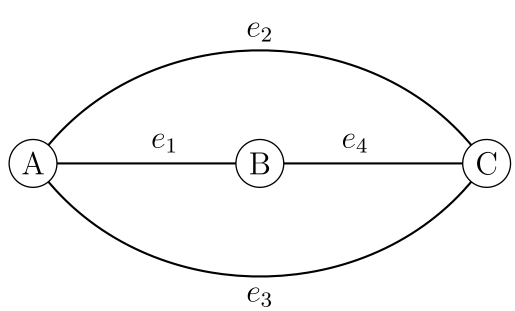
As we saw last week, graphs can be a great tool to methodically analyze and solve complex problems. This week, we will be looking at some tools that can be used to analyze graphs.

Properties of Graphs

Two vertices of a graph are called *adjacent*, if they are connected by an edge. Two edges of a graph are called *incident*, if they share a vertex. Also, a vertex and an edge are called *incident*, if the vertex is one of the two the edge connects.

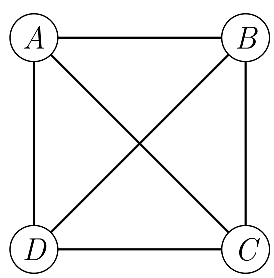
Consider a graph with m vertices, $v_1, v_2, \ldots, v_m$ and n edges, $e_1, e_2, \ldots, e_n$. The *incidence matrix* of the graph is an $m \times n$ table of numbers T organized the following way. In the case the vertex v_i is incident to the edge e_j that is not a loop, $T_{ij} = 1$. If e_j is a loop, $T_{ij} = 2$. $T_{ij} = 0$ otherwise. For example, on the right hand side below is the incidence matrix of the graph on the left.

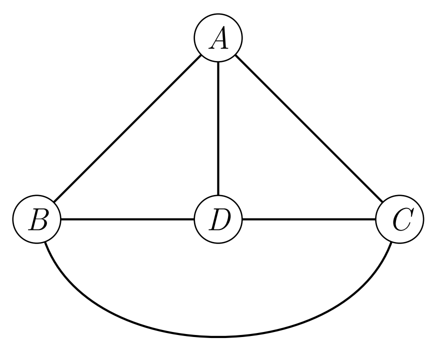


$$T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

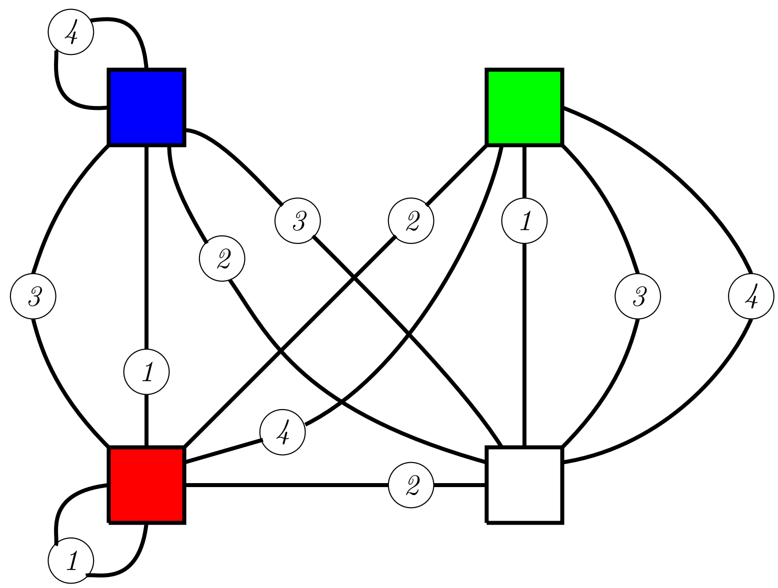
Problem 1 *Looking at the incidence matrix T above, it is not hard to notice that the sum of the entries in every column equals 2. Would it always be the case for an undirected graph? Why or why not?*

Problem 2 *Write down the incidence matrices T_1 and T_2 for the graphs below. Do the matrices tell you that you are looking at two different pictures of one and the same graph? Why or why not?*



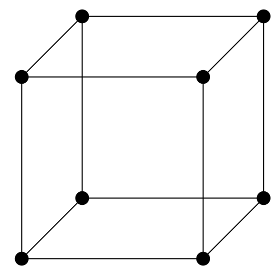


Problem 3 *Write down the incidence matrix of the Instant Insanity configurations' graph from week 1, which is reproduced below.*



The *degree* $d(v)$ of a vertex v of a graph is the number of the edges of the graph incident to the vertex.

Problem 4 *Do all the vertices of a cube, considered as a graph, have the same degree? If so, what is the degree?*



Theorem 1 *For any graph, the sum of the degrees of the vertices equals twice the number of the edges.*

Problem 5 *Prove Theorem 1.*

Problem 6 *Prove the following corollary of Theorem 1. The number of vertices of odd degree in any graph is even.*

Problem 7 *One girl tells another, "There are 23 kids in my class. Isn't it funny that each of them has 7 friends in the class?" "This cannot be true," immediately replies the other girl. How did she know?*

Problem 8 *In a small European country, each city is connected to other cities of the country by five roads. There are 25 inter-city roads in the country. How many cities are there?*

A *path* in a graph is a subgraph having the following property. Its vertices can be renumbered $v_1, v_2, \ldots, v_n, v_{n+1}$ so that $e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, \ldots, e_n = \{v_n, v_{n+1}\}$. In other words, a path is a subgraph that can be drawn without lifting the pen off the paper. A *cycle* is a closed path, i.e. a path such that $v_1 = v_{n+1}$.

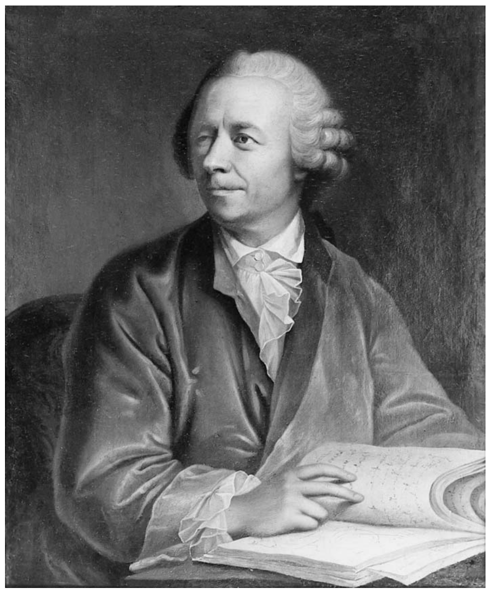
A graph is called *connected* when there is a path between every pair of its vertices. A graph is called *disconnected* otherwise.

Problem 9 *Draw a graph with four vertices, all of degree one.*

Problem 10 *By the year 2050, Hyperloop has developed the following routes: Portland – New York City, New York City – Boston, Boston – Los Angeles, Los Angeles – San Francisco, San Francisco – New York City, San Francisco – Portland, San Diego – Washington DC, San Diego – Austin, Austin – Charlotte, Charlotte – Washington DC, Austin – Washington DC, and Los Angeles – New York City. Each route has high-speed passenger and cargo pods travelling both ways. Is it possible to get from Los Angeles to Washington DC by a Hyperloop pod? Why or why not?*

Eulerian paths and cycles

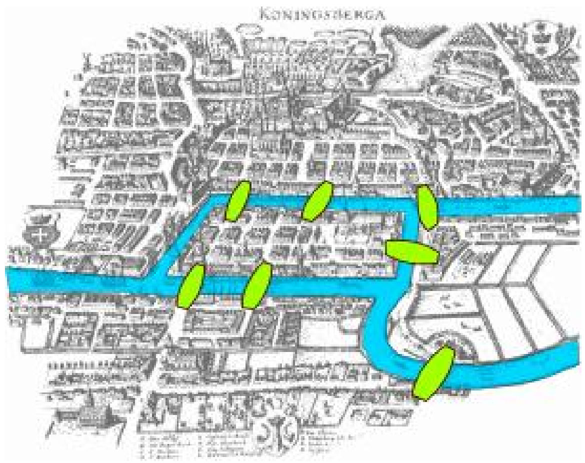
An *Eulerian path* is a path in an (undirected) graph that traverses each edge exactly once. An *Eulerian cycle* is a closed Eulerian path. They are named in honour of a great Swiss mathematician, Leonhard Euler (1707-1783), considered by many as the founding father of the graph theory.



Leonhard Euler

The seven bridges of Königsberg problem

During his stay in the city of Königsberg, then the capital of Prussia, Euler came up with, and solved, the following problem. Can one design a walk that crosses each of the Königsberg's seven bridges once and only once? The picture of Königsberg of Euler's time is provided below.



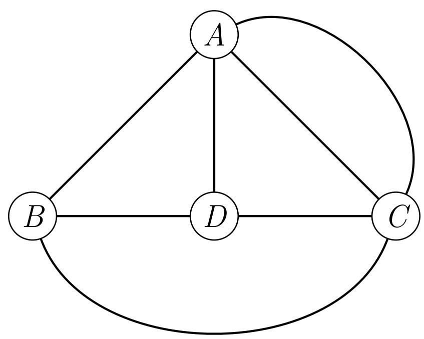
Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges.

Problem 11 *Draw a graph with the vertices corresponding to the landmasses from the picture above and with the edges corresponding to the Königsberg's seven bridges. What are the degrees of each of the graph's vertices?*

Problem 12 *Was it possible to design a Eulerian walk in the city of Königsberg at the time of Euler? Why or why not?*

Problem 13 *Can you add a bridge so a walk becomes possible?*

Problem 14 *Find a Eulerian path in the following graph.*



Problem 15 *Does the above graph contain a Eulerian cycle? If not, how many edges do you need to add to create an Eulerian cycle?*

Problem 16 *What is the minimal number of times you need to lift the pencil off the paper to draw a cube without repeating any edge?*

Challenge Problems

Problem 17 *Can you add a bridge to the map of Königsberg without creating an Eulerian walk?*

Problem 18 *Given any graph, how can you tell whether an Eulerian cycle is possible? (Hint: Think about the degrees of all the vertices.)*