ORMC Advanced **July 10, 2022**

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Warm-up

Problem 1 *How many positive integers* n *between* 1 *and* 100 *(i.e.* $1 \leq n \leq 100$*) are there such that* $\sqrt{n}$ *is a rational number?*

Problem 2 *Find*

$$\sum_{i=1}^{n} \log\left(\frac{i+1}{i}\right).$$

in terms of n*.*

Problem 3 *Find a four digit number which is a square of an integer, and such that its first two digits are the same and its last two digits are the same. (Example: if* 1122 *were the square of an integer, which it is not, it would satisfy the condition about the first two digits being the same and its last two digits being the same).*

Non-Integers in Binary

Continuing on from where we left off, let us recall that in Base 10, we represent *decimals* with a decimal point: $0.3 = 3 \cdot 10^{-1}$. Naturally, we can do the same in any base.

Problem 4 *Find the decimal representations of the following binary numbers. (Recall: the bar over a number represents a repeated decimal, eg.* $0.0\overline{1} = 0.01111111\ldots$*, but* $0.\overline{01} = 0.0101010101\ldots$*)*

$$0.1_2 =$$

$$0.01_2 =$$

$$0.\overline{1}_2 =$$

$$0.\overline{01}_2 =$$

Problem 5 *Find the binary representations of the following decimal numbers.*

$$\frac{1}{8}_{10} =$$

$$\frac{1}{3}_{10} =$$

$$\frac{1}{7}_{10} =$$

Problem 6 *You may notice that some fractions repeat forever and some fractions end after a finite number of digits. Come up with a conjecture as to when fractions of the form* $\frac{1}{p}$ *repeat in binary and when they do not. Discuss with the other students and see if you can agree on a conjecture.*

Problem 7 *Prove your conjecture. Give an explanation for every statement you make. You are encouraged to work together on this problem. (Hint: consider the long division algorithm.)*

Other Bases

Consider base 16 (also known as hexadecimal). There are more than 10 digits, so we need more than the traditional $0-9$ to represent a number in base 16. The most common substitutes are the letters $A-F$, giving us the 16 digits as $\{0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}$ in ascending order (thus, $A_{16}=10_{10}$).

Problem 8 *Find the binary representations of the following hexadecimal numbers (without converting to decimal).*

$$A_{16} =$$

$$1F_{16} =$$

$$C0FFEE_{16} =$$

Problem 9 *Find the hexadecimal representations of the following binary numbers (without converting to decimal).*

$$1010_2 =$$

$$100100111001_2 =$$

$$10010101001010100101010_2 =$$

Example 1 *Let us investigate a method to convert between binary and hexadecimal that is very efficient. Let us first convert hexadecimal to binary. We want to find* $1FCC_{16}$ *in binary. Let us recall that* $1FCC_{16} = 1 \cdot 16^3 + F \cdot 16^2 + C \cdot 16^1 + C \cdot 16^0$, *because of how the place-value system works. But because* 16 *is a power of* 2, *we can rewrite this as* $1 \cdot 2^{12} + F \cdot 2^8 + C \cdot 2^4 + C \cdot 2^0$. *We can rewrite each hexadecimal digit in binary:* $1_{16} = 0001_2$, $2_{16} = 0010_2, \ldots, F_{16} = 1111_{16}$. *Then we get*

$$1FCC_{16} = 0001 \cdot 2^{12} + 1111 \cdot 2^8 + 1100 \cdot 2^4 + 1100 \cdot 2^0.$$

But we know that $2^{12} = 1000000000000_2$, *and the long multiplication here is quite easy:* $0001 \cdot 2^{12} = 0001000000000000_2$. *Thus, our result is*

$$\begin{aligned} 1FCC_{16} &= 0001000000000000_2 \\ &\quad +111100000000_2 \\ &\quad +11000000_2 \\ &\quad +1100_2 \\ &= 0001111111001100_2 \end{aligned}$$

Problem 10 *Come up with a method to go from base* 2 *to base* 16.

Problem 11 *If I give you a number in binary, what bases can you convert this number into using this method? What about if I give you a number in ternary (base* 3*)?*

Non-Integers in Other Bases

Problem 12 *Find the ternary (base 3) representations of the following decimal numbers.*

$$\frac{1}{8}_{10} =$$

$$\frac{1}{3}_{10} =$$

$$\frac{1}{7}_{10} =$$

Problem 13 *Let* p *be a positive integer. What will* $\frac{1}{p-1}$ *look like in base* p*? What will* $\frac{1}{p+1}$ *look like in base* p*? (Hint: use long division).*

Challenge Problems

Problem 14 *Let p and q be positive integers greater than 1. Let $\frac{1}{p}$ be represented in base q. Prove that the number of digits in the repeating part (eg. if $q = 10$ and $p = 60$, then $\frac{1}{p} = 0.01\overline{6}$, so the repeating part is 6) is strictly smaller than p. For the purpose of this question, numbers such as 0.125 are considered to have repeating part 0 (ie. $0.125 = 0.125\overline{0}$).*

Problem 15 *Prove that $\displaystyle\sum_{i=1}^{\infty} \frac{r-1}{r^i} = 1$ for any positive integer $r > 1$. (Hint: consider what this fraction will look like in base r).*