Inequalities

The Arithmetic Mean-Geometric Mean (AM-GM) inequality is the following:

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq (x_1 x_2 \ldots x_n)^{1/n}, \quad x_1, x_2, \ldots, x_n > 0.$$

The quantity on the left is probably familiar as the *arithmetic mean*, or average, of the numbers $x_1, x_2, \ldots, x_n$, while the quantity on the right is the *geometric mean*.

Exercises:[1]

1. Prove the AM-GM inequality for $n = 2$, that is:

$$\frac{a+b}{2} \geq \sqrt{ab}, \quad a, b \geq 0.$$

2. Prove the AM-GM inequality for $n = 4$, which we will suggestively write

$$\frac{\frac{x_1+x_2}{2} + \frac{x_3+x_4}{2}}{2} \geq (x_1 x_2 x_3 x_4)^{\frac{1}{4}}, \quad x_1, x_2, x_3, x_4 \geq 0.$$

   Can you see how to prove the AM-GM inequality for $n = 2^k$? General n? ($\star$)

3. Prove the Geometric Mean-Harmonic Mean (GM-HM) inequality:

$$(x_1 x_2 \ldots x_n)^{1/n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

Here are more inequalities to prove. The AM-GM inequality may prove useful in some cases:

1. Prove that $x + \frac{1}{x} \geq 2$ for any $x > 0$
2. Prove that $1 + x \geq 2\sqrt{x}$ for any $x \geq 0$
3. Prove that $\frac{x^2+y^2}{2} \geq xy$ for any x, y.
4. Prove that $x^2 + y^2 + z^2 \geq xy + yz + xz$ for any x, y, z. (Hint: Use a previous problem.)
5. Prove that $2(x^2 + y^2) \geq (x + y)^2$ for any x, y.
6. Prove that $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$ for any $x > 0$, $y > 0$.
7. Prove that $x^2 + y^2 + 1 \geq xy + x + y$ for any x, y.
8. Prove that $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$ for any $x, y, z > 0$.
9. Prove that $x^4 + y^4 + z^4 \geq xyz(x + y + z)$ (Hint: Use a previous problem.)

[1] Problems from Fomin, Genkin, Itenberg, *Mathematical Circles (Russian Experience)*, Universities Press