Vectors

Prepared by Mark on June 20, 2022.
Based on a handout by Oleg Gleizer.

Part 1: Warm-Up

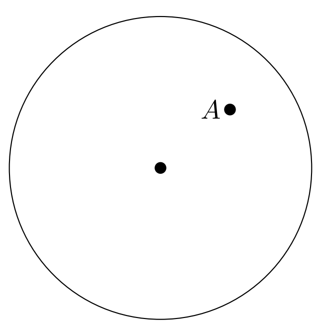
Problem 1:
Simplify the following fraction:

$$\frac{\frac{1}{2}}{\frac{1}{3}-\frac{1}{4}} =$$

Problem 2:
Simplify the following fraction:

$$\frac{\frac{a}{b}-\frac{c}{d}}{\frac{a}{d}+\frac{c}{b}} =$$

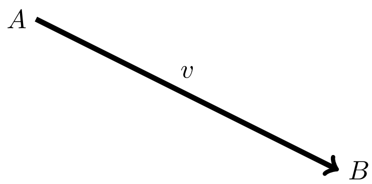
Problem 3:
The point A is placed inside a circle.



Cut the circle into two parts so that you can move one to make a circle centered at A.

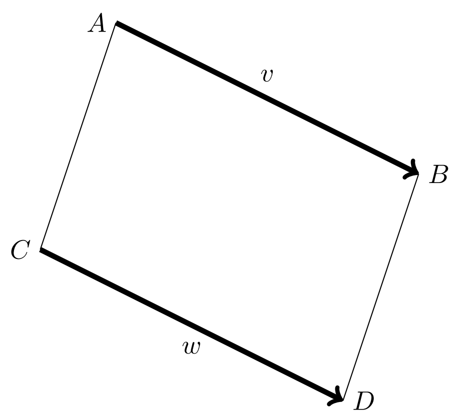
Part 2: Vectors

Definition 1:
A vector in the Euclidean plane is a directed line segment.



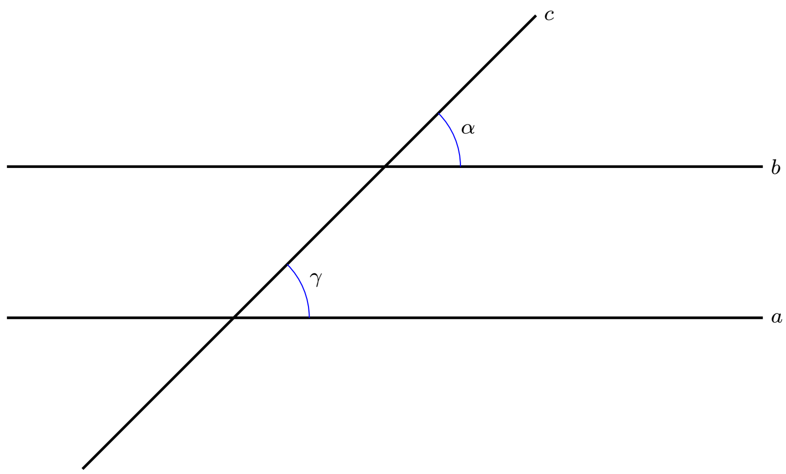
$v = \overrightarrow{AB}$

For the vector $v = \overrightarrow{AB}$, point A is called *initial* and point B is called *terminal*.
Two vectors, $v = \overrightarrow{AB}$ and $w = \overrightarrow{CD}$ are considered equivalent if the quadrilateral $ABDC$ is a parallelogram.



In other words, two vectors are equivalent if they have the same length and direction. If this is the case, we write $v = w$.
Note 1: Convince yourself that this is true. Why are these two definitions of vector equivalence interchangable?
Note 2: A vector is characterized by its direction and length. One cannot make a formal definition out of this observation, because a "direction" is formally defined in terms of a vector.

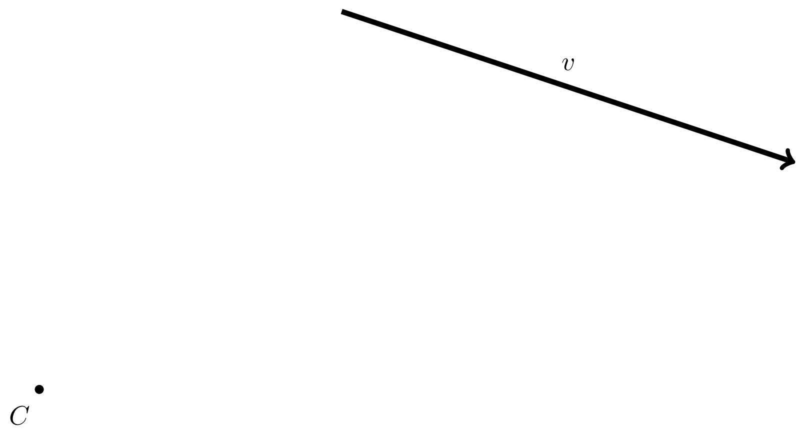
Theorem 2:
If two distinct straight lines in the Euclidean plane form the angles of equal size with a third straight line in the plane, then they are parallel.
In other words, to check that the lines a and b on the picture below have no common point, you don't need to travel to infinity. All you need to do is to measure the angles α and γ. If $\alpha = \gamma$, then a is parallel to b.



Euclid's 5th postulate:
For any straight line in the (Euclidean) plane and for any point away from it, there exists a unique straight line that passes through the point and is parallel to the original line.

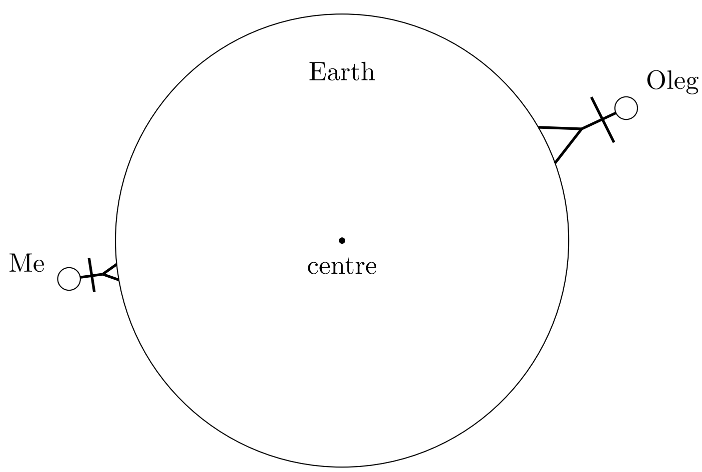
Problem 4:
Use a compass and a ruler to draw a straight line parallel to the one given below and passing through the given point not lying on the original straight line.

Problem 5:
Use a compass and a ruler to construct a vector w with initial point C equal to the vector v below.



Physical forces, such as the force of gravity or the force that pulls together two magnets are vectors in three dimensions. The direction of a vector shows the direction in which the corresponding force is acting. The length of the vector shows the strength of the force.

Problem 6:
On the picture below, draw the vectors of the gravitational pull the Earth exerts on you and on your Math Circle leader.



- Where do the gravitational force vectors point? Why?
- If Oleg is twice as heavy as you are, how would you draw the gravitational pull vectors on the above picture?

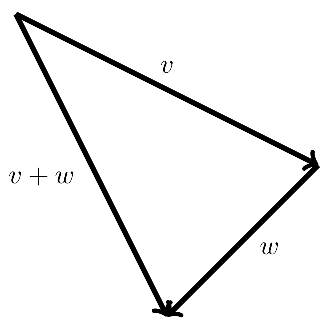
Motion can be represented by a vector, too. The direction of the velocity vector shows the direction in which an object is moving at the moment. The length of the velocity vector represents the object's *speed*. It shows how fast an object is moving at the moment.

Problem 7:
The truck on the picture below is going 30 mph. The car on the same picture is speeding at 90 mph the opposite way. Draw the corresponding velocity vectors.

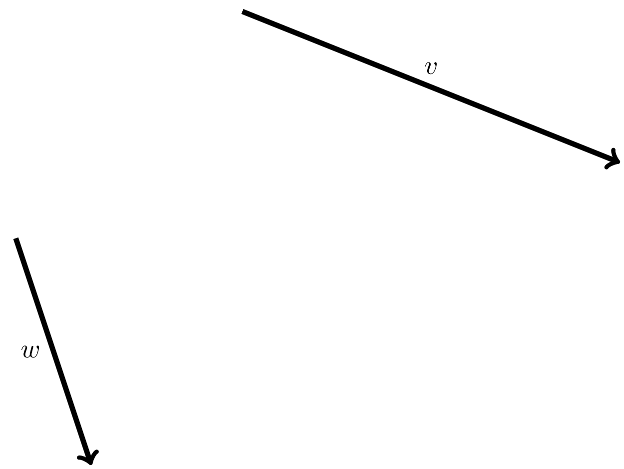
Velocities and forces of the real world are vectors in three dimensions. To keep things simple, we'll begin our study of vectors in two dimensions. Everything we are going to learn about vectors in two-dimensional space is be valid in a Euclidean space of three—or more—dimensions.

Part 3: Adding Vectors

To find the sum of two vectors v and w, one needs to take w so that the initial point of w coincides with the terminal point of v. The vector originating at the initial point of v and terminating at the terminal point of w is the sum $v + w$.



Problem 8:
Use a compass and a ruler to construct the sum $v+w$ of the vectors v and w given below.

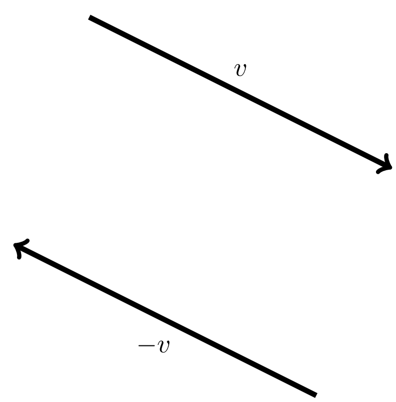


Definition 3: The Zero Vector
A vector that has coinciding initial and terminal points is called the *zero vector* and is denoted as $\overrightarrow{0}$. According to the above definition of the vector addition,

$$v+\overrightarrow{0}=\overrightarrow{0}+v=v \tag{1}$$

for any vector v.

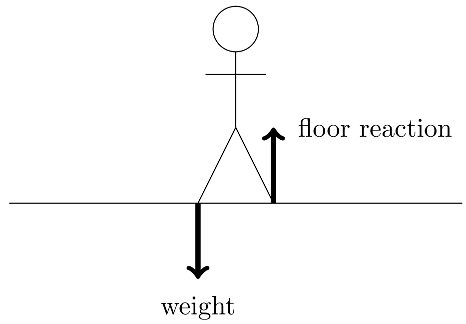
Definition 4: Inverses of Vectors
A vector w such that $w+v=\overrightarrow{0}$ is called the *inverse* of v and is denoted as $-v$. The vector $-v$ lies either on the same straight line as v or on a parallel one, has the same length as v, but points in the opposite direction:



Note that $-v+v=\overrightarrow{0}$ by definition, but the validity of the equation $v+(-v)=\overrightarrow{0}$ follows from the definitions of vector addition and the zero vector. We can combine both into the following:

$$-v+v=v+(-v)=\overrightarrow{0} \tag{2}$$

Here is an important example of an inverse vector. When you stand still, the floor pushes you up with the force opposite to the force of the gravitational pull, a.k.a. *weight*.



The two opposing vectors add up to the zero vector, and therefore you don't move.

Problem 9:
Give an example of a different pair of opposite forces.

The following is the last thing we'll mention about the opposite vectors. The formula $w - v$ is defined as $w + (-v)$ for any vectors v and w:

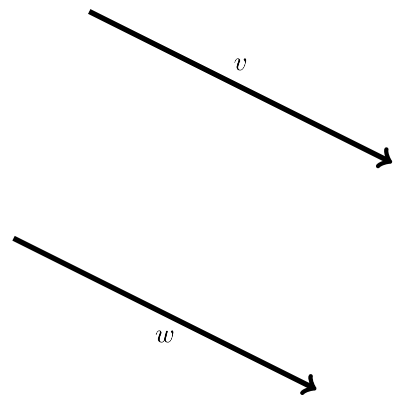
$$w - v = w + (-v) \tag{3}$$

Problem 10: Dividing a segment into parts

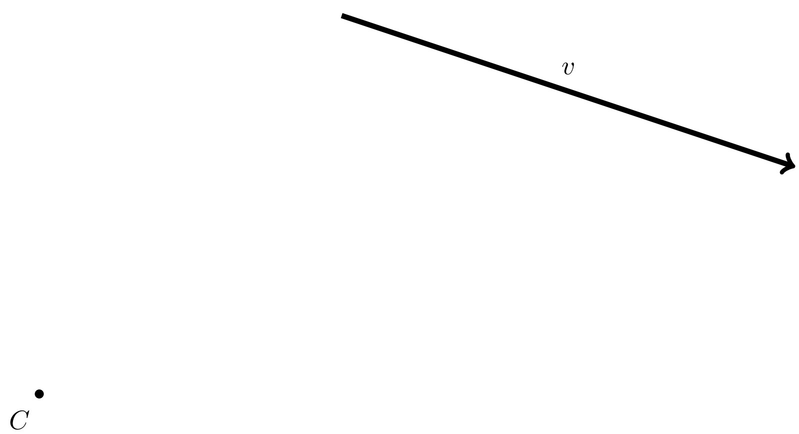
Using your compass and ruler, divide a segment into three equal parts. How would you split it into four? five? Have an instructor check your answer before moving on.

Hint: Don't skip this problem, you'll need it later. Make sure you check your answer!

Problem 11:
Is it possible to check whether $v = w$ on the picture below using only a compass? Why or why not?

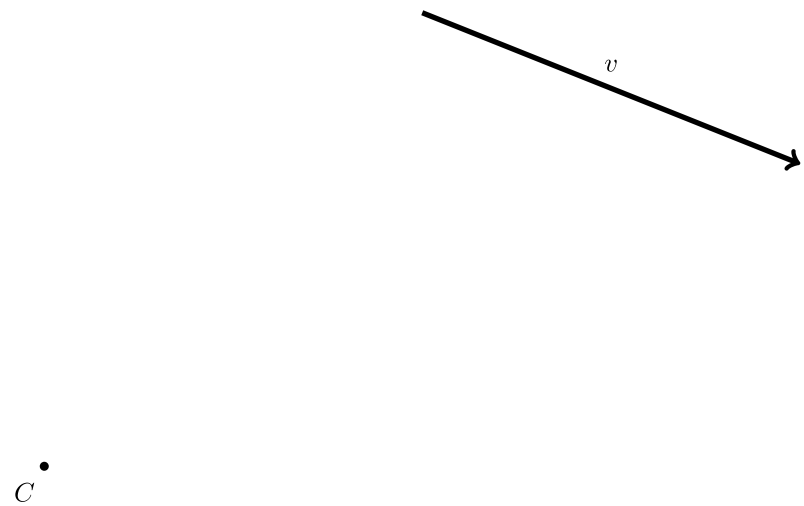


Problem 12:
Use a compass and a ruler to construct the vector $w = -.75v$ for the vector v given below such that point C is its terminal point.



Note: With the tools we have thus far, we can multiply vectors by any rational number using only a compass and a ruler. Multiplying a vector by an irrational number is a bit more tricky, but it is doable...

Problem 13:
Use a compass and a ruler to construct the vector $w = \sqrt{3}v$ for the vector v given below so that C is its initial point.
Hint: Pythagoras.



Part 4: Bonus

Problem 14: Oldaque de Freitas' Puzzle
Two ladies are sitting in a street café, talking about their children. One lady says that she has three daughters. The product of the girls' ages equals 36 and the sum of their ages is the same as the number of the house across the street. The second lady replies that this information is not enough to figure out the age of each child. The first lady agrees and adds that her oldest daughter has beautiful blue eyes. The second lady then solves the puzzle. Please do the same.

Problem 15: There must be a better way...
Using pen and paper, sum up all the integers from 1 to 1000.