Week 1: Instant Insanity Puzzle

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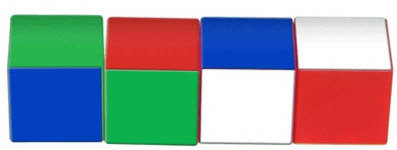
Introduction

Instant Insanity is a popular puzzle, created by Franz Owen Armbruster, currently marketed by the *Winning Moves* company, and sold, among other places, on *Amazon.com*. It is advisable to have the puzzle in front of you before reading this worksheet any further.

The puzzle consists of four cubes with faces colored with four colors, typically red, blue, green, and white. The objective of the puzzle is to stack the cubes in a row so that each side, front, back, upper, and lower, of the stack shows each of the four colors.



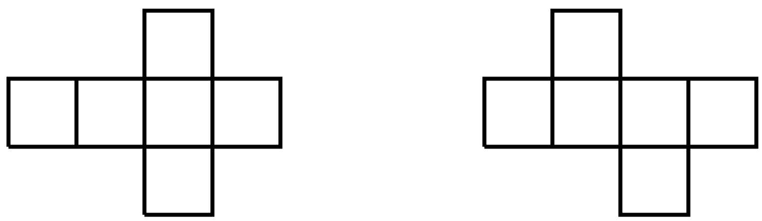
There exist 41,472 different arrangements of the cubes. Only one is a solution. Finding this one by trial and error seems about as likely as winning a lottery jackpot. However, we have witnessed a few ORMC students doing just that. Those were some truly extraordinary children!

Problem 1 *Try to solve the puzzle.*

To approach a task this formidable, the more ordinary people, like the authors of this worksheet, need to forge some tools.

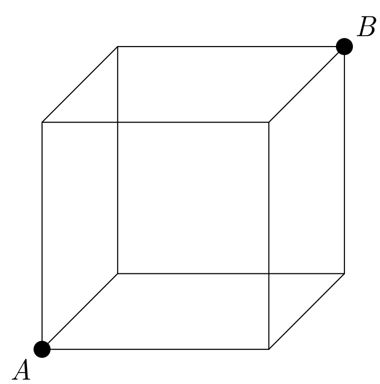
Cubic nets

A *cubic net* is a 2D picture simultaneously showing all the six sides (a.k.a. faces) of a 3D cube, please take a look at the examples below.



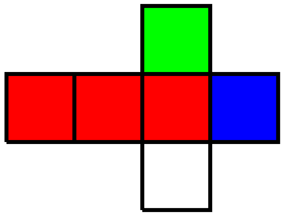
Problem 2 *Draw a cubic net different from the two above.*

Problem 3 *An ant wants to crawl from point A of a cubic room to the opposite point B, please see the picture below.*

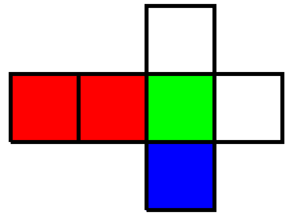


The insect can crawl on any surface, a floor, ceiling, or wall, but cannot fly through the air. Find at least two different shortest paths for the ant (there is more than one). Hint: use a cubic net.

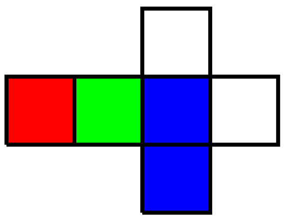
Now we have the means to take a better look at the cubes from the puzzle, the cubic nets!



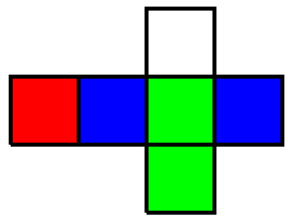
Cube 1



Cube 2



Cube 3



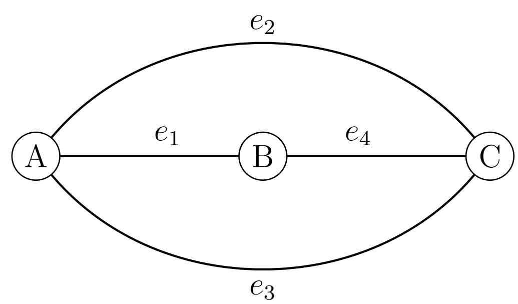
Cube 4

We can see that all the four cubes of the puzzle are different. Cube 1 is the only one having three red faces. Cube 2 uniquely possesses two red faces. Cube 3 is the only one having two adjacent blue faces. Cube 4 also has two blue faces, but they are opposite to each other. Finally, Cube 4 is the only one having two green faces.

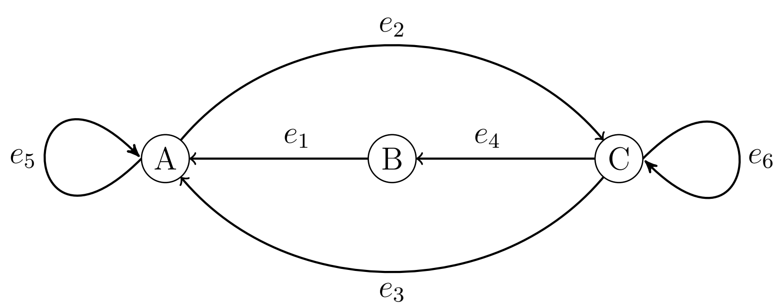
Cubic nets are great for visualizing a single cube, but they are not as efficient at describing various configurations of all the four of them. We need one more tool.

Graphs

A *graph* is a set of vertices, $\mathcal{V} = \{v_1, v_2, \ldots\}$, connected by edges, $\mathcal{E} = \{e_1, e_2, \ldots\}$. If an edge e connects the vertices v_i and v_j, then we write $e = \{v_i, v_j\}$. If the order of the vertices does not matter, the graph is called *undirected*. Typically, the word *graph* means an undirected graph. A graph is called a *directed graph*, or a *digraph*, if the order of the vertices does matter. For example, the (undirected) graph below has three vertices, A, B, and C, and four edges, $e_1 = \{A, B\}$, $e_2 = \{A, C\}$, $e_3 = \{A, C\}$, and $e_4 = \{B, C\}$.



An edge connecting a vertex to itself is called a *loop*. For example, the digraph below has two loops, $e_5 = (A, A)$ and $e_6 = (C, C)$, in addition to the edges $e_1 = (B, A)$, $e_2 = (A, C)$, $e_3 = (C, A)$, and $e_4 = (C, B)$.



Note that we use different notations for an edge of a graph and digraph. An edge of a graph, $e = \{A, B\}$, is a set of the two vertices it connects. In this case, the order does not matter, $\{A, B\} = \{B, A\}$ as sets. An edge of a digraph, $e = (A, B)$ is a list (an ordered set) of the vertices it connects. The order does matter now, $(A, B) \neq (B, A)$.

The endpoint of a directed edge e is called its *head* and denoted $h(e)$. The starting point on an edge e is called its *tail* and denoted $t(e)$. For example, $h(e_4) = B$ and $t(e_4) = C$ for the digraph above.

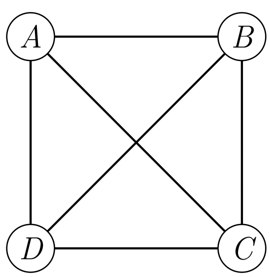
In the book, we use the letters $\mathcal{V}$ and $\mathcal{E}$ for the sets of vertices and edges in a graph. We use the letters V and E for the numbers of the vertices and edges. In other words, V is the number of elements in the set $\mathcal{V}$, E is the number of elements in the set $\mathcal{E}$.

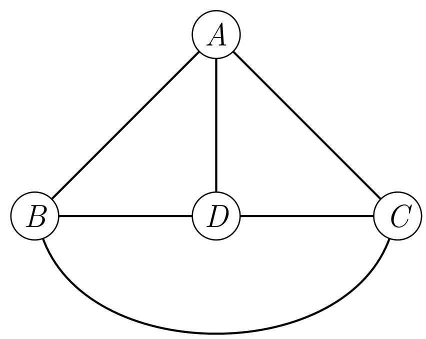
Problem 4 *Given a graph with V vertices and E edges that has no loops, how many ways are there to orient the edges so that the resulting digraphs are all different?*

Problem 5 *Draw an undirected graph that has the vertices A, B, C, D, and E and the edges $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{A, E\}$, $\{B, C\}$, $\{C, D\}$, and $\{D, E\}$.*

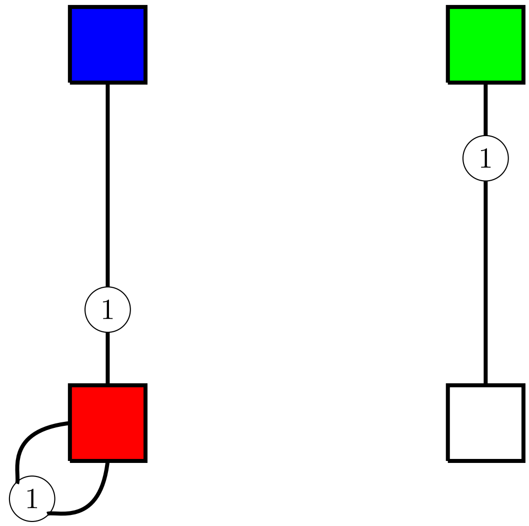
Two different pictures of a graph can look very dissimilar.

Problem 6 *Prove that the two pictures below represent the same graph by comparing the sets of their vertices and edges.*

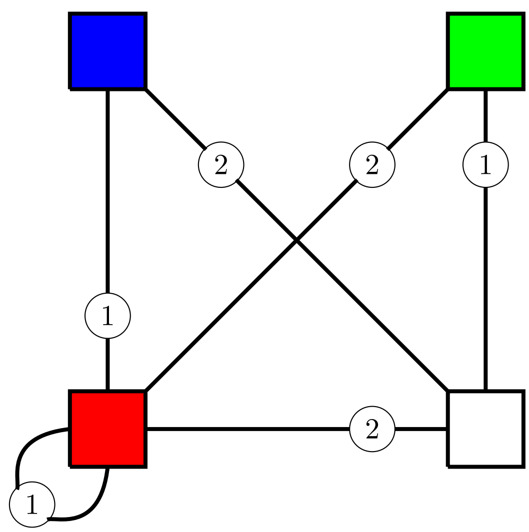




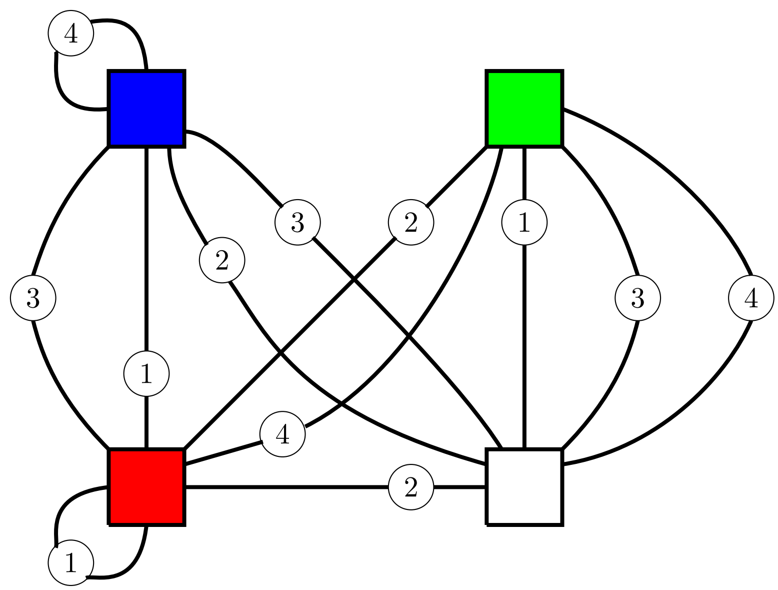
Getting back to the puzzle, let us represent Cube 1, see page 3, by a graph. The vertices will be the face colors, Blue, Green, Red, and White, $\mathcal{V} = \{B, G, R, W\}$. Two vertices will be connected by an edge if and only if the corresponding faces are opposing each other on the cube. Cube 1 has the following edges, $e_1 = \{B, R\}$, $e_2 = \{G, W\}$, and the loop $e_3 = \{R, R\}$. To emphasize that all the three edges represent the first cube, let us mark them with the number 1.



Cube 2 has the following pairs of opposing faces, $\{B, W\}$, $\{G, R\}$, and $\{R, W\}$. Let us add them to the graph as the edges e_4, e_5, and e_6.



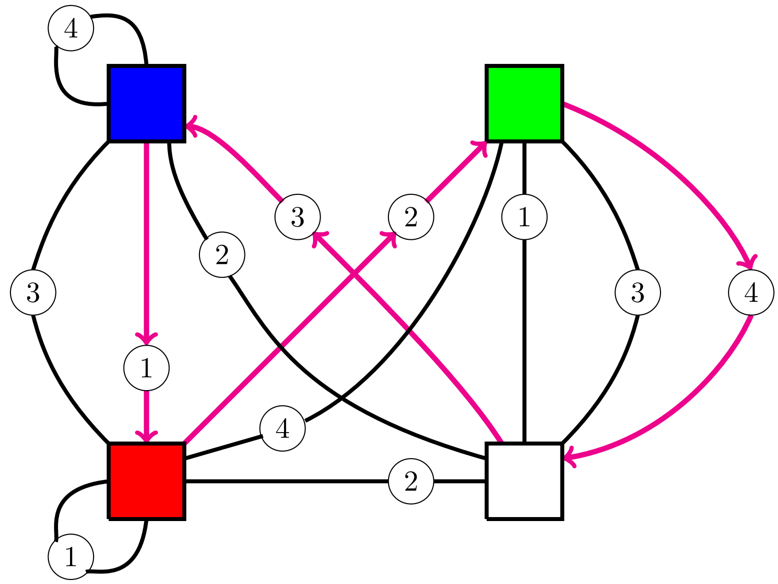
Let us now make the graph represent all the four cubes.



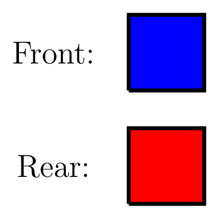
Problem 7 *Check if the above representation is correct for Cubes 3 and 4.*

With the help of the above graph, solving the puzzle becomes as easy as a walk in the park, literally. Imagine that the vertices of the above graph are the clearings and the edges are the paths. An edge marked by the number i represents two opposing faces of the i-th cube. Let us try to find a closed walk, a.k.a. a cycle, in the graph that visits each clearing once and uses the paths marked by the different numbers, $i = 1, 2, 3, 4$. If we order the front and rear sides of the cubes accordingly, then the front and rear of the stack will show all the four different colors in the order prescribed by our walk.

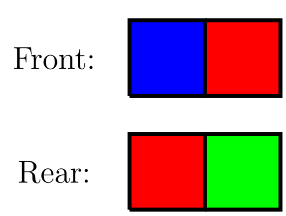
For example, here is such an (oriented) cycle, represented by the magenta arrows on the picture below.



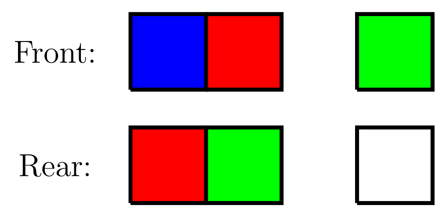
The first leg of the walk tells us to take Cube 1 and to make sure that its blue side is facing forward. Then the red side, opposite to the blue one, will face the rear.



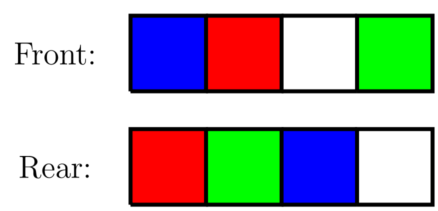
The next leg of the walk tells us to take Cube 2 and to place it in such a way that its red side faces us while the opposing green side faces the rear. Since we go in a cycle that visits all the colors one-by-one, neither color repeats the ones already used on their sides of the stack.



The third leg of the walk tells us to take Cube 4, not Cube 3, and to place it green side forward, white side facing the rear.



Finally, the last leg of the walk tells us to take Cube 3 and to place it the white side facing forward, the opposite blue side facing the rear.



Now the front and rear of the stack are done. If we manage to find a second oriented cycle in the original graph that has all the properties of the first cycle, but uses none of its edges, we would be able to do the upper and lower sides of the stack and to complete the puzzle. Using the edges we have already traversed during our first walk will mess up the front-rear configuration, but there are still a plenty of the edges left!

Problem 8 *Complete the puzzle.*