ORMC Intermediate 2 Spring Competition

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Welcome to the last week of this quarter's math circle! The tournament will consist of n questions, and your team will be given two of them at a time and will not be given any additional ones until you correctly solve or give up on one of them. The questions are a mix of computational questions and short proofs. You will be awarded 1 point for solving a problem first try, 0.5 points second try, and 0 points if you exhaust both tries and/or give up on a problem.

Problem 1. Three runners start running simultaneously from the same point on a standard 400 meter track. They each run in the same direction at speeds of $4.4, 4.8, 5.5$ meters per second, respectively. The runners all stop once they are all together again. How many seconds are they running for?

Problem 2. How many solutions are there to the equation

$$x^2 \equiv 5 (\text{mod } 11)$$

Problem 3. Find a mixed Nash equilibrium in the following two-player zero-sum game:

	L	R
U	5	-2
D	-2	0

Problem 4. In a round-robin tournament with 6 teams, each team plays one game against each other team, and each game results in one team winning and one team losing. At the end of the tournament, the teams are ranked by the number of games won. What is the maximum number of teams that could be tied for the most wins at the end of the tournament?

Problem 5. Alice owns 100 plots of land, in a row from east to west. One day, she is notified that there is gold on one plot of her land, but not of where the gold is located. She has a metal detector which can indicate whether the gold is to her west or east (or if it happens to be directly where she is standing). However, the metal detector doesn't have much battery, so she can only use it a few times. Find the minimum number of uses needed, and also give an algorithm to find the gold in that number of uses.

Problem 6. A solid cube of side length 1 is removed from each corner of a solid cube of side length 3. How many edges does the remaining solid have?

Problem 7. Two perpendicular lines intersect at the point $A = (6, 8)$. Their y-intercepts, P and Q, add up to zero. What is the area of ΔAPQ?

Problem 8. Let a, b be relatively prime integers where $a > b > 0$. Suppose that

$$\frac{a^3 - b^3}{(a-b)^3} = \frac{73}{3}$$

Find $a - b$.

Problem 9. Recall that last quarter we studied the **depth-first search algorithm**. Suppose we have a graph G with V vertices and E edges. The algorithm is as follows:

1. Choose a vertex of G, called the node, and add it to a subgraph H.
2. Starting at a vertex, travel along an edge to any vertex not previously visited. Add both the edge and the vertex to H.
3. Repeat Step 2 until a vertex is reached whose neighbors have all been previously visited. Return to the node when this occurs.
4. Repeat Steps 2 and 3 until all neighbors of the node have been visited.

What is this algorithm's runtime?

Problem 10. How many ways can 2021 be written as the sum of exactly two prime numbers?

Problem 11. A square of area 4 is inscribed in a square of area 5. Each vertex of the smaller square divides a side of the larger square into two segments of length a and b. Find ab.

Problem 12. How many solutions are there to the equation

$$x^2 + 2x \equiv 30 (\text{mod } 103)$$

Problem 13. Every day at school, Bob climbs a flight of 8 stairs. He can take $1, 2$, or 3 steps at a time. How many different ways can he climb the stairs?

Problem 14. Which of the following polynomials are solvable in radicals?

$$x^4 + 3x^3 - 15x + 12, \;\; x^5 - 2, \;\; x^5 - 101x + 101$$

Problem 15. A poker deck contains only three cards, a King (K), and Queen (Q), and a Jack (J). The cards are ordered $K > Q > J$, with the higher card winning if the cards are shown. Two players are playing against each other with 1 in the pot (to be won), and are each dealt one card uniformly randomly without replacement. Suppose player 1 has already checked (chose not to bet). Player 2 may then bet (let's say they may only bet 1), in which case Player 1 may call or fold (see previous handout or ask the instructors for rule clarifications), or player 2 may check, which ends the round. Compute a Nash equilibrium for this simplified poker game.

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?