Olympiad Group Spring Week 6: Power of a Point and Radical Axis

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Introduction

Recall the following from previous weeks:

Definition (Cyclic Quadrilaterals): If quadrilateral $ABCD$ is cyclic (i.e. can be inscribed in a circle), then:

(i) $\angle ABC + \angle ADC = 180°$,

(ii) $\angle BAC = \angle BDC$,

and so on for the other angles of the quadrilateral. It is also true that given points A, B, C, D, if one of the above is satisfied, then we can conclude that $ABCD$ is a cyclic quadrilateral.

Theorem (Power of a Point): Let ω be a circle, and P a point. Let a line through P intersect ω at A, B, and let another line through P intersect ω at C, D. Then, $PA \cdot PB = PC \cdot PD$.

Now, consider the following new definitions:

Definition (Power of a Point): Let ω be a circle with center O and radius r, and let P be a point. Then, the power of P with respect to ω is defined as $\text{Pow}_\omega(P) = OP^2 - r^2$.
Note: This is related to the previous theorem by: $PA \cdot PB = |\text{Pow}_\omega(P)|$ (see problem 7).

Definition (Radical Axis): Given two circles ω_1 and ω_2, the radical axis of ω_1 and ω_2 is the set of points P which have equal power with respect to both circles, i.e. P such that $\text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_2}(P)$. This set of points is in fact a straight line.
Note: if the circles intersect, it is then the line through the points of intersection. (Why?)

Theorem (Radical Center): Given three circles, their pairwise radical axes all intersect at a single point (as long as the centers of the circles are not collinear).

Another fact worth noting is that the converse of Power of a Point Theorem also holds: Given 4 points A, B, C, D, with P defined as the intersection of lines AB and CD, if $PA \cdot PB = PC \cdot PD$, then A, B, C, D are concyclic (that is, $ABCD$ is a cyclic quadrilateral). You may use this fact in one of the below problems.

Problems

1. In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. Find AP.

2. Let ABC be an equilateral triangle with side length 1 and let ω be its circumcircle. Some chord $\overline{PQ}$ of the circle is split into three equal parts by $\overline{AB}$ and $\overline{AC}$. Find PQ.

3. Two circles of radii 17 centered at A and B intersect at X and Y. If $XY = 30$, find AB.

4. Let ω_1 and ω_2 be two circles intersecting at X and Y. Suppose that the common external tangent of the two circles touches ω_1 at A and ω_2 at B. If XY intersects AB at M, prove that M is the midpoint of AB.

5. A circle has center $(-10, -4)$ and radius 13. Another circle has center $(3, 9)$ and radius $\sqrt{65}$. The line passing through the two points of intersection of the two circles has equation $x + y = c$. What is c?

6. In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

7. Given a circle ω and point P, let a line through P intersect the circle at A, B. Prove that $PA \cdot PB = |\text{Pow}_\omega(P)|$. Hint: Let M be the midpoint of AB; then note that OM is perpendicular to line PA, creating helpful right triangles.

   The absolute value is simply to account for $\text{Pow}_\omega(P)$ being negative if P is inside ω. Also, note how this implies the previously written Power of a Point Theorem.

8. Let ABC be a triangle, and let H be its orthocenter. If H' is the reflection of H over line BC, prove that $ABH'C$ is a cyclic quadrilateral.

9. (Extension of Radical Center Theorem) Let ω_1 and ω_2 be two circles. Select points A and B on ω_1 and points C and D on ω_2. Then, A, B, C, D are concyclic if and only if lines AB and CD intersect on the radical axis of ω_1 and ω_2.

10. Let ABC be a triangle, let M be the midpoint of BC, let E be the foot of the altitude from B to AC, and let F be the foot of the altitude from C to AB. Prove that ME, MF, and the line through A parallel to BC are all tangent to (AEF).

11. Let ABC be an acute triangle with $BC = 48$. Let M be the midpoint of BC, and let D and E be the feet of the altitudes drawn from B and C to AC and AB respectively. Let P be the intersection of the line through A parallel to BC and line DE. If $AP = 10$, compute the length of PM.

12. Let C be a point on semicircle Γ with diameter AB. Let D be the midpoint of arc AC. Let E be the foot of the altitude from D to BC, and let F be the intersection of Γ and AE. Prove that BF bisects DE.

13. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

14. In acute triangle ABC, angle B is greater than C. Let M be the midpoint of BC. D and E are the feet of the altitude from C and B respectively. K and L are the midpoints of ME and MD respectively. If KL intersects the line through A parallel to BC at T, prove that $TA = TM$.